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Univalent functions–selected topics, by Glenn Schober, Lecture Notes in Mathematics, no. 478, Springer-Verlag, Berlin, Heidelberg, New York, 1975, iv + 200 pp.

The term univalent applied to a mapping means simply that it is one-to-one. However the combination "univalent functions" has a much more specific meaning, referring to regular (holomorphic) or meromorphic functions which determine one-to-one mappings. They may be considered in various domains of definition, even on a Riemann surface, but attention is often directed to certain specific classes; two of the most important are denoted by S and Σ. The first consists of functions $f(z)$ regular and univalent for $|z| < 1$ with Taylor expansion about the origin

$$z + \sum_{n=2}^{\infty} A_n z^n. \tag{1}$$

The second consists of functions $f(z)$ meromorphic and univalent for $|z| > 1$ with Laurent expansion about the point at infinity

$$z + \sum_{n=0}^{\infty} c_n z^{-n}. \tag{2}$$

The theory of univalent functions had its beginnings in results of Koebe obtained in 1907 and 1909 which may be stated as follows.

I. *There exists an absolute constant κ such that for $f \in S$ the function values $w = f(z)$ for $|z| < 1$ fill the circle $|w| < \kappa$ where κ is the largest value for which this is true.*

II. *There exist positive quantities $m_1(r)$, $M_1(r)$ depending only on r such that for $f \in S$, $|z| = r$,*

$$m_1(r) \leqslant |f(z)| \leqslant M_1(r).$$

III. *There exist positive quantities $m_2(r)$, $M_2(r)$ depending only on r such that for $f \in S$, $|z| = r$,*

$$m_2(r) \leqslant |f'(z)| \leqslant M_2(r).$$

These results aroused great interest and a number of people began to work in the field. In particular, the first consistent method was introduced, the area method, and used by Gronwall, Bieberbach and Faber. This method utilizes the simple fact that the area enclosed by the image under $f \in \Sigma$ of the circle $|z| = r$ $(r > 1)$ is positive. Expressing this in terms of the coefficients of f it is easy to obtain the Area Theorem:

$$\sum_{n=1}^{\infty} n|c_n|^2 \leqslant 1.$$

From this one can easily prove that $\kappa = \frac{1}{4}$ in I and obtain the precise expressions for $m_1(r)$, $M_1(r)$, $m_2(r)$, $M_2(r)$ in II and III. Further, Bieberbach proved that, for $f \in S$, $|A_2| \leqslant 2$ and conjectured that, in general, $|A_n| \leqslant n$. Since it was made in 1916 this conjecture has been a standing challenge and has inspired, directly or indirectly, many of the subsequent developments in the field. While it has been verified for a few initial values, for general n only less precise estimates have been obtained.

In the intervening years a number of more sophisticated methods have been introduced: Löwner's parametric method, the method of the extremal metric, the method of contour integration and the variational method. For a brief description of these methods we refer to the reviewer's book *Univalent functions and conformal mapping*. It should be remarked that Golusin resumed the study of the area method in connection with multivalent functions (the same convention being understood as for the term univalent) and quite recently a more sophisticated version of the area method has proved very fruitful, two slightly different forms being inspired by the papers of N. A. Lebedev and the reviewer. All of these methods are essentially geometric, that is, they concentrate on the aspect of a univalent function as a mapping rather than on its representation in analytic forms.

Another line of investigation has been to study subclasses of the family S. The two most familiar such subclasses are K (convex functions), consisting of the functions in S mapping $|z| < 1$ onto a convex domain (in these notations we follow Schober's symbols) and S^* (starlike functions), consisting of the functions in S mapping $|z| < 1$ onto a domain starlike with respect to the origin (a domain D is said to be starlike with respect to the origin if for every point w_0 of D the segment joining w_0 to the origin lies in D). These families have an important connection with the family P of functions f regular in $|z| < 1$ with $\mathfrak{R}f > 0$ and $f(0) = 1$. Other families worthy of note are the real univalent functions S_R consisting of those functions in S with all A_n real, the typically real functions T_R consisting of functions $f(z)$ regular in $|z| < 1$ normalized as in (1) with $(\mathfrak{I}z)\,(\mathfrak{I}f(z)) \geqslant 0$ for all z in $|z| < 1$ and the close-to-convex functions C consisting of functions $f(z)$ regular in $|z| < 1$ normalized as in (1) with $\mathfrak{R}[f'(z)/e^{i\alpha}\varphi'(z)] > 0$, for some real α and $\varphi \in K$. The standard method of treating these families is to use the geometric properties to derive a characteristic analytic condition from which further properties are derived.

Quite recently a number of authors, with Kühnau probably the first, have studied the subclasses S_K and Σ_K of functions of S or Σ which admit a K-quasiconformal extension to the complement of $|z| < 1$ or $|z| > 1$. In case a mapping is differentiable the condition for it to be K-quasiconformal is that the ratio of maximal to minimal directional distortion at each point is at most K $(K > 1)$. In case the mapping is not required to be differentiable there are a number of characterizations, all equivalent but too technical for the space at our disposal.

Schober's book is entitled *Univalent functions–selected topics*. It is in no sense a general treatment of univalent functions but the various topics included have, for the most part, a unifying theme: the book might have been subtitled *What functional analysis can do for univalent functions*. At first glance families of univalent functions would not seem very appropriate subjects for the application of functional analysis since they are not linear spaces in any natural sense. However, there are several ways to proceed. Many of the particular function classes above are closely related to P which is a convex subset of a linear space. Again the family of univalent functions in a domain can be regarded as a subset of the family of functions regular in that domain (with any appropriate normalizations). The latter becomes a topological vector space when limits are understood in the sense of uniform convergence on compact subsets.

The first major selection of topics utilizes the methods of convexity theory and extreme points. An extreme point of a subset A of a linear space is one which admits no nontrivial representation $\lambda x + (1 - \lambda)y$, $x, y \in A$, $\lambda \in (0, 1)$. The set of extreme points of A is denoted by E_A. The key result is then that if A is a compact subset of a locally convex linear topological space X and if x' is a continuous linear functional on X, then $\max_A \Re x' = \max_{E_A} \Re x'$. In the first chapter the author, following F. Holland, by an explicit manipulation identifies the extreme points of the set P as a subset of the space of functions regular in the unit circle. They are indeed the functions $(1 + \eta z)/(1 - \eta z)$, $|\eta| = 1$. A theorem of Choquet then implies the Herglotz representation: if $f \in P$ there exists a unique nonnegative Borel measure μ (probability measure) such that

$$f(z) = \int_{|\eta|=1} (1 + \eta z)/(1 - \eta z)\, d\mu, \quad \int_{|\eta|=1} d\mu = 1.$$

Of course the procedure could be reversed, the Herglotz representation obtained in the usual way and the extreme points deduced from it. Chapter II contains applications of this and similar results to the families S^*, K, S_R, T_R, C and several additional families. In particular, $f \in S^*$ if and only if there exists a probability measure μ such that

$$f(z) = z \exp\left[-2 \int_{|\eta|=1} \log(1 - \eta z)\, d\mu\right].$$

This is used to obtain various properties of starlike functions, for example, their extreme points and the sharp bound for their coefficients in the representation (1), namely $|A_n| \leqslant n$. Similar results are given for the other classes. In particular, there is given an inequality for convex functions due to Ruscheweyh and Sheil-Small from which they deduced the proof of the Pólya-Schoenberg Conjecture: if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in K$, $g(z) = \sum_{n=0}^{\infty} b_n z^n \in K$, then $\sum_{n=0}^{\infty} a_n b_n z^n \in K$ (as the author, following their work, does in Chapter III).

In the context of the above function families the methods of convexity and extreme points provide a very nice unified exposition of the results, including many originally derived by other methods. The success basically rests on the fact that the required extreme points can be explicitly determined and are not too numerous. It is not difficult to find other contexts where one or the other condition fails and the role of extreme points for certain extremal problems is much less incisive.

The second major selection of topics attaches to the consideration of continuous linear functionals on the space of functions regular on a domain D (notation $H(D)$, functionals $H'(D)$) regarded as a topological vector space as indicated above. The primary result is an old, but not sufficiently well-known, result of Caccioppoli characterizing such functionals: for a domain D in the complex plane $\mathcal{C}$, $L \in H'(D)$, there exists a function g regular in $\mathcal{C} - D$ (i.e., extending to an open set O_g containing $\mathcal{C} - D$ and regular in each component thereof) vanishing at ∞ and a finite system of rectifiable Jordan curves C in $D \cap O_g$ such that for $f \in H(D)$,

$$L(f) = (2\pi i)^{-1} \int_C f(z)g(z)\,dz.$$

This is used in two contexts. In one the standard normalization of univalent functions at a point (as in expansion (1)) is replaced by normalization assigning the values of two linear functionals. Further the problem of maximizing $\Re L$ for such a linear functional on a family so normalized is considered.

In the first instance the main question is when a family so normalized is compact. A fairly elementary sufficient condition is obtained which includes some of the desired standard results but not others, for example, the compactness of Σ', the subset of Σ with $c_0 = 0$ in (2), for which it is necessary to revert to the standard proof. A partial converse is also obtained. For some such families it is shown that there is an analogue of Brickman's result on extreme points for the family S: the corresponding functions map the unit disc onto a domain bounded by a monotone arc (i.e., one which meets each circle centre the origin at most once). There is also given a number of geometric properties of functions satisfying this condition.

In dealing with the problem of maximizing $\Re L$ on a family $\mathcal{F}$ of univalent functions normalized by two linear functionals (and assumed to be compact) the essential technique is to obtain a variation of functions within the family $\mathcal{F}$. A necessary condition for extremality then takes the form that a linear functional depending on an extremal function is nonnegative. This is manifested for a number of elementary variations and, in Chapter 10, for the Schiffer boundary variation. The latter leads to conditions of the form that the image domain under an extremal function is bounded by trajectory arcs of a quadratic differential (in this context this means arcs satisfying a differential equation $r(w)(dw/dt)^2 = 1$ with a rational function $r(w)$ and a suitable parameter t). A number of known results for extremal functions for $\Re A_n$ for $f \in S$ are extended to this situation under suitable conditions, including the fact that if a boundary component contains a zero of the quadratic differential it is rectilinear.

Some similar problems are considered for nonlinear functionals. This leads, in particular, to the existence of certain canonical conformal mappings. (In the standard usage a conformal mapping is just the same as a univalent function but perhaps the use of distinct terms is justified as representing somewhat different viewpoints.) A representative result is that for any plane domain D containing the point at infinity and for $z_0 \in D$ there exists a function g univalent in D with the expansion (2) at infinity, $g(z_0) = 0$ and mapping D onto a domain bounded by radial slits. Nothing is said about the corresponding uniqueness results and, indeed, this method seems little suited to treat them. (They are very significant for domains of infinite connectivity.) There are also derived a number of results more usually obtained by generalizations of the area method.

The third major selection of topics deals with the families S_K, Σ_K and certain related families. While in the preface the author characterizes the treatment as variational it is very definitely a functional analytic sort of variation as opposed to the distinctly geometric type of variation first attempted by Schiffer for these classes which encountered serious technical difficulties. This technique is used to derive some general remarks on variational problems for functions with quasiconformal extensions and to treat a number of explicit bounds for the initial coefficients in their series expansions. The author does not delineate the exact extent to which these bounds have been pushed but the most sophisticated result given here appears to be the bound for $|A_2|$ in the expansion (1) for $f \in S_K$. There are also derived some results analogous to those usually associated with the area method for functions in S and Σ. The area method leads to a very simple proof of $|A_4| \leqslant 4$ for $f \in S$ but the analogous technique is much less successful here. This reinforces the suspicion that the success in the former case is somewhat accidental.

Overall the book collects and expounds a variety of results which would not otherwise be accessible except in the original memoirs. The exposition is generally very careful and the author makes an effort to supply at least a survey of material utilized from other sections of mathematics. The brief presentation of results from convexity theory in Appendix A gives a good feeling for the material and makes the applications in the text seem natural. The survey of results for quasiconformal mappings in Chapter 12 is much less successful. The reviewer feels that it fails to provide the intuitive feeling that would be so useful in the following chapters. If the presentation throughout the book has a weakness it is in a striving for what seems occasionally like excessive generality. It is true that this sometimes manifests new facets not present in the more familiar cases but sometimes the treatment of a more special case first would have made it easier to recognize what is going on. Specifically the beginning of Chapter 13 is at times pretty murky and only in reading the applications of Chapter 14 are some of the concepts clarified. However this tendency does not detract substantially from the interest and usefulness of the book. It should stimulate activity in a number of the questions presented.

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