constant coefficient hypoelliptic equations is not mentioned. There is no discussion of, or references for, questions of hypoellipticity or local solvability of general equations, noncoercive boundary value problems, nonlinear versions of any results or methods, pseudodifferential or Fourier integral operators. While it would be unreasonable to expect more than a very brief discussion or passing reference for most of these omissions, it is unfortunate not to have that much.

The lament of the previous paragraph is that a very good text is not still better. In his preface, Treves cites two aims: "recalling the classical material to the modern analyst, in a language he can understand," and "exploiting the [classical] material, with the wealth of examples it provides, as an introduction to the modern theories." Anyone sympathetic to these aims would do well to read the entire preface, and the book.

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Simple Noetherian Rings, by John Cozzens and Carl Faith, Cambridge Tracts in Mathematics, no. 69, Cambridge Univ. Press, New York and London, 1975, xvii + 135 pp., $12.95.

This book is concerned with a class of rings which are "simple" only in a standard technical sense. Speaking descriptively, it would be much more appropriate to entitle this material *Complicated Noetherian Rings*. Technically, a simple ring is a nonzero ring R (associative with unit, as far as this book is concerned) in which the only two-sided ideals are the two trivial ones, 0 and R. (When dealing with rings without unit, one assumes in addition that R does not have zero multiplication, i.e., $R^2 \neq 0$.) The most basic class of simple rings consists of the division rings. Although the structure of division rings is already enormously complex, one considers the division rings to be "known" in the context of general rings, and tries to relate the structure of larger classes of rings to the class of division rings in various ways. In order to be able to say much at all about simple rings in general, some chain condition is usually imposed, such as the artinian condition (any descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of one-sided ideals is ultimately constant, i.e., $I_n = I_{n+1} = \cdots$ for some n) or the noetherian condition (any ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of one-sided ideals is ultimately constant).

The first (and most widely used) general structure theorem for simple rings is of course the Wedderburn-Artin Theorem: *Any simple artinian ring R is isomorphic to the ring of all $n \times n$ matrices over some division ring D, and both n and D are uniquely determined by R.* Alternatively stated, this theorem says that R is isomorphic to the endomorphism ring of a finite-dimensional vector space over D. Because of the Hopkins-Levitzki Theorem, which states that every artinian ring is also noetherian, the Wedderburn-Artin Theorem characterizes a portion of the class of simple noetherian rings. That not all simple noetherian rings fall into this portion is shown by examples such as the "Weyl algebras" $A_1(K)$: For any field K, $A_1(K)$ is the K-algebra with two generators x and y subject to the sole relation $xy - yx = 1$. This algebra is always noetherian but never artinian, and in characteristic zero it is simple as well.

The study of simple noetherian rings in general did not begin until after the work of Goldie and Lesieur-Croisot in the late 1950's. Their pivotal result is that any prime noetherian ring R (i.e., a noetherian ring in which the product of any two nonzero two-sided ideals is nonzero) can be embedded in a simple artinian ring Q in such a way that every element of Q is expressible as a fraction whose numerator and denominator both belong to R. (In this situation, R is said to be an "order" in Q. The noetherian hypothesis here, which was used by Lesieur-Croisot, is stronger than necessary, and Goldie's Theorem gives the appropriate necessary and sufficient conditions under which R is an order in a simple artinian ring.) The simple artinian ring Q is of course the ring of all $n \times n$ matrices over a division ring D, and R may be roughly approximated by a matrix ring over an order in D, as shown by the Faith-Utumi Theorem: *R may be embedded in Q in such a way as to contain the ring of all $n \times n$ matrices over an order F in D* (*however, F need not have a unit*).

These results in particular provide a representation of any simple noetherian ring as a subring of a matrix ring over a division ring. An alternative representation, analogous to the second form of the Wedderburn-Artin Theorem mentioned above, is due to Faith: *Any simple noetherian ring is isomorphic to the endomorphism ring of a finitely generated projective module over an Ore domain* (i.e., *a not necessarily commutative integral domain in which any pair of nonzero elements have a nonzero common multiple*). However, the Ore domain obtained need not be simple, although it may be chosen to have at most one nontrivial two-sided ideal. Stronger results are obtained in the special case covered by Goldie's Principal Right Ideal Theorem: *Any prime principal right ideal ring is isomorphic to the ring of all $n \times n$ matrices over some right noetherian integral domain* (*although this integral domain need not be a principal right ideal domain*).

The theorems just discussed form the backbone of the theory of simple noetherian rings. While many other structural results are known, they are more technical, and require more terminology than is comfortable to introduce here. In addition, the theory of simple noetherian rings includes studies of certain classes of examples, and of certain restricted classes of simple noetherian rings.

Two examples of simple noetherian rings have been studied in greatest detail, namely differential polynomial rings and skew polynomial rings. Starting with a ring R and a derivation δ on R (i.e., an additive endomorphism of R which satisfies the usual product rule for derivatives), the "differential polynomial ring" over R and δ (better described as a ring of formal linear differential operators) is constructed additively as polynomials over R in an indeterminate θ, with multiplication induced by the relation $\theta a = a\theta + \delta(a)$

(for all $a \in R$). For example, if R is the polynomial ring in one variable over a field K and δ is the ordinary formal derivative on R, then the differential polynomial ring is just the Weyl algebra $A_1(K)$. Suitable other choices of R and δ also result in simple noetherian rings. A similar construction yields skew polynomial rings, starting from a ring R and a ring endomorphism ρ of R. In this case, the construction again uses the additive group of polynomials over R in an indeterminate t, but multiplication is induced from the relation $ta = \rho(a)t$ (for all $a \in R$). It is also possible to combine these examples in a construction which uses a ring endomorphism and a modified derivation simultaneously.

In his thesis, Cozzens constructed differential polynomial rings which are simple principal ideal domains over which every simple module is injective. Also, he and Osofsky (separately) constructed additional examples of such rings as localizations of skew polynomial rings. These examples gave new impetus to the study of what are now called V-rings, i.e., rings for which every simple module is injective. (A much earlier result of Kaplansky showed that a commutative ring is a V-ring exactly when it is von Neumann regular.) These examples also belong to (and generated the study of) the class of PCI-rings: rings R for which every proper cyclic module (i.e., every cyclic module not isomorphic to R) is injective.

Readers who have perservered through this long an outline of simple noetherian ring theory are now entitled to an indication of how the book under review treats this theory. There are seven chapters, of which the first four deal with the most general parts of the theory. The first chapter is concerned with the machinery used to study endomorphism rings of projective modules. It covers Morita's Theorem (on equivalence of module categories), and Faith's "Correspondence Theorem", which relates the ideals of the endomorphism ring of a projective module P (not necessarily a generator) to the submodules of P and to the ideals contained in the trace ideal of P. Chapter 2 initiates the structure of simple noetherian rings, covering for example some of the consequences of the existence of a uniform right ideal, and proving Faith's representation of simple noetherian rings as endomorphism rings of projective modules over Ore domains. Also included are short discussions of the phenomena which occur in the cases of homological dimensions 0, 1, 2. Chapter 3 develops the differential polynomial and skew polynomial rings used for various classes of examples. The fourth chapter is devoted to orders in simple artinian rings. It includes Goldie's Theorem, the Faith-Utumi Theorem, and Goldie's Principal Right Ideal Theorem, as well as a bit of the general theory of maximal orders. Chapters 5 and 6 develop the structure of V-rings and PCI-rings, respectively, and Chapter 7 briefly discusses fourteen open problems.

Most book reviews bemoan the prevalence of misprints, minor errors and omissions in proofs, and the usual few major errors. This book rates about average on that score. Interestingly, the errors seem to *decrease* exponentially as one proceeds through the book. There are a few errors to be wary of in the statements of propositions, as follows. In Proposition 1.16, V must be finitely generated over B_0 in order to conclude that $U = U_0 B$. Only one direction ($\Leftarrow$) of Lemma 2.5 is valid, and for this implication, V need not be faithful or finitely generated. The implication (2) $\Rightarrow$ (3) in Proposition 2.26, and the corresponding statement in Lemma 2.27, are false. (There exist orders in a commutative field which are isomorphic as rings but not equivalent as orders.) In Theorem 2.28, either I or K (in addition to J) must be essential in A for the conclusion to hold.

In their preface, the authors state that their book is intended for the "general reader of mathematics". Such a "general reader", however, should be well-practiced in manipulating rings and modules upside down and backwards (in addition to right and left) in order to follow the proofs here, particularly in the first chapter, where the results deal simultaneously with a ring, a module, endomorphism and biendomorphism rings of the module, and various duals of the module. Innumerable minor points about rings and modules are used without mention, the justifications given for many points are irrelevant or incorrect, and at times proofs require facts not discussed until succeeding chapters. Later in the book, many of the proofs resemble close paraphrases of the original research papers, and in fact, over eight pages of Chapter 3 are reproduced directly from one of Cozzens' articles. Also, the proofs here are less complete than earlier in the book, and occasionally outside references are quoted for key points. Thus only readers with a fair degree of proficiency in noncommutative ring theory would be likely to find their way through the complications, gaps, and errors in this material. Experts in noncommutative ring theory will of course find this book easy enough to skim through, but not to read in detail. In particular, anyone attempting to use this book as a text should expect to review each detail of each proof before each lecture.

As a reference, this book deserves credit for bringing together results which were previously rather scattered in the literature. The major theorems of course appear in a number of other sources, but this book seems to be the first attempt to collect any sizable portion of simple noetherian ring theory in one place. (The book's usefulness in this regard is enhanced by the form of the introduction, which discusses in outline the main results of the book.) However, this collation of disparate sources seems to have proceeded at the expense of the exposition. On one hand, major results such as Morita's Theorem, Goldie's Theorem, and the Faith-Utumi Theorem are each more clearly developed in several other sources. On the other hand, the more specialized material on V-rings and PCI-rings is presented in virtually the same form as the original research papers, with hardly any attempt to expand the dense proofs.

In conclusion, to average all these observations into a single inadequately expressive final grade, I would recomment LP: libraries should purchase this book so that it will be available for occasional reference.

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