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# CONTINUITY OF THE KOBAYASHI METRIC IN DEFORMATIONS AND FOR ALGEBRAIC MANIFOLDS OF GENERAL TYPE[1]

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Let $M$ be a complex manifold and $TM$ the holomorphic tangent bundle of $M$. The disc of radius $r$ in $\mathbf{C}$ will be denoted by $\Delta(r)$, and $\Delta$ will stand for $\Delta(1)$. The Kobayashi pseudo-distance $d_M$ and its infinitesimal pseudo-metric $F_M$ are defined as follows:

(i) If $p, q \in M$, then

$$d_M(p, q) = \inf_{\{a_i\} \subset \Delta} \frac{1}{2} \sum_i \log \frac{1 + |a_i|}{1 - |a_i|}$$

where the infimum is over all finite sets $\{a_i\} \subset \Delta$ such that there exist $n$ analytic mappings $f_i$: $\Delta \longrightarrow M$ for which $f_1(0) = p, f_i(a_i) = f_{i+1}(0)$ for $i = 1, n-1$, and $f_n(a_n) = q$.

(ii) If $\langle x, \xi\rangle \in TM$, then $F_M(x, \xi) = \inf 1/R$ where the infimum is over all $R$ such that there exists an analytic $f$: $\Delta(R) \longrightarrow M$ with $f_x(0, \partial/\partial z|_0) = \langle x, \xi\rangle$.

Royden has shown [5] that $d_M(p, q) = \inf_\sigma \int_\sigma F(\sigma, \dot{\sigma})$ where the infimum is over all piecewise smooth curves from $p$ to $q$.

The manifold $M$ is said to *hyperbolic* if $d_M(p, q) \neq 0$ whenever $p \neq q$.

A deformation of $M$ is specified by giving an analytic space $S \subset \mathbf{C}^k$ and a family of integrable almost complex structures $\{\varphi_s | s \in S\}$ on $M$ such that $\varphi_o = 0$ for some point $o \in S$; each $\varphi_s$ is therefore a $C^\infty$ $TM$-valued (0, 1) form on $M$, satisfying $\bar{\partial}\varphi_s - [\varphi_s, \varphi_s]/2 = 0$. See [2] for details. Using $\varphi_s$, we can construct a bundle isomorphism $\Phi_s$: $TM \longrightarrow TM_s$, where $TM_s$ is the holomorphic tangent bundle for the complex structure given by $\varphi_s$. Set $F_{M_s} = F_s$. Assume that $o = 0$, the origin in $\mathbf{C}^k$.

THEOREM A. *Given $\langle x, \xi\rangle \in TM$ and $\epsilon > 0$, there exists a $\delta > 0$ such that if $|s| < \delta$ then $F_s(y, \eta) \leq F_o(x, \xi) + \epsilon\|\xi\|$ for all $\langle y, \eta\rangle$ in a neighborhood of $\langle x, \Phi_s\xi\rangle$ in $TM_s$. (Here $\|\xi\|$ is the norm provided by a coordinate system.)*

This basic upper semicontinuity result can be improved if $F_M$ is known to

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be continuous on $TM$; e.g., if $F_M$ is continuous and $M$ is compact, the $\delta$ can be chosen to be independent of $\langle x, \xi \rangle$.

THEOREM B. *If $M$ is compact and hyperbolic, then $F_s(y, \eta)$ is continuous on $\bigcup_{s \in U} TM_s$ and $d_s(p, q)$ is continuous on $U \times M \times M$ for $U$ any sufficiently small neighborhood of $o \in S$.*

This theorem follows from Theorem A and the result of R. Brody [1] that $F_s$ is lower semicontinuous in $s$ for $s$ sufficiently close to 0 when $M$ is hyperbolic.

Using Theorem B and the Kuranishi theory of versal deformations [2], we obtain the following result about moduli of compact hyperbolic manifolds. See [3] for a similar result for manifolds with ample canonical bundle.

THEOREM C. *Let $M$ be a compact hyperbolic manifold and let $\mathfrak{M}$ denote the collection of isomorphism classes of hyperbolic complex structures on the underlying differentiable manifold of $M$. Then $\mathfrak{M}$ has the structure of a Hausdorff complex space such that if $\{M_s\}_{s \in S}$ is any family of hyperbolic complex structures on $M$, then the map sending $s$ to the isomorphism class of $M_s$ is a morphism from $S$ to $\mathfrak{M}$.*

Examples of Royden (unpublished) show that $F_M$ is not always lower semicontinuous on $M$.

DEFINITION. A projective algebraic manifold $M$ is said to be of general type if

$$\limsup_{m \to +\infty} \frac{1}{m^n} H^0(M, \mathcal{O}(K^m)) > 0.$$

Here $K$ denotes the canonical bundle and $\dim_{\mathbf{C}} M = n$. Let $\eta$ denote the Kobayashi-Eisenman pseudo-volume for $M$ [4].

THEOREM D. *If $M$ is compact algebraic and there exist sections $S_0, \ldots, S_k$ of $H^0(M, \mathcal{O}(K^m))$ which provide a projective embedding of $M$ such that $S_i\bar{S}_i/\eta$ is bounded for every $i = 0, \ldots, k$, then $F_M$ is continuous on $TM$.*

COROLLARY. *If $M$ is projective algebraic of general type then $F_M$ is continuous on $TM$.*

Proofs and details of the above will appear in [6].

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