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BOUNDED MEAN OSCILLATION WITH ORLICZ NORMS AND DUALITY OF HARDY SPACES

BY JAN-OLOV STRÖMBERG

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This paper is a summary of the contents in [5] (with the same title).

BMO was introduced by John and Nirenberg as the space of locally integrable functions f on $\mathbf{R}^n$ such that

$$\int_Q |f(x) - f_Q|\, dx \leqslant Am(Q) \tag{1}$$

for some constant f_Q and for all cubes Q in $\mathbf{R}^n$, where $A \geqslant 0$ only depends on f. (m is the Lebesgue measure in $\mathbf{R}^n$.) John and Nirenberg [2] have shown that (1) will imply

$$m\{x \in Q;\ |f(x) - f_Q| > t\} \leqslant C_1 m(Q) \exp(-C_2 t/A), \quad t > 0, \tag{2}$$

with A the same as in (1).

If we assume that f a priori only is a measurable function on $\mathbf{R}^n$ and replace (1) by

$$m\{x \in Q;\ |f(x) - f_Q| > A'\} < \tfrac{1}{2} m(Q), \tag{1'}$$

then it is still true that $f \in BMO$ and (1) holds with $A \leqslant CA'$. In fact, a proof similar to that of [2] shows that $(1)'$ implies (2). If the constant $1/2$ in $(1)'$ is replaced by a larger number, the result is no longer true.

Let us now assume that the numbers A and A' in (1) resp. $(1)'$ are not uniform but depend on x (but not on Q containing x). We define $M^{\#}f(x)$ resp. $M_0^{\#}f(x)$ as the infimum of such $A(x)$ resp. $A'(x)$. (2) will now be replaced by

$$\begin{aligned} &m\{x \in Q;\ |f(x) - f_Q| > t\} \\ &\qquad \leqslant C_1 m(Q) \exp(-C_2 t/s) + m\{x \in Q;\ M_0^{\#}f(x) > s\}, \quad s, t > 0. \end{aligned} \tag{3}$$

From (3) it follows that $\|M_0^{\#}f\|_{L^p}$ is equivalent to $\|M^{\#}f\|_{L^p}$, $1 < p < \infty$. It was shown by Fefferman and Stein in [1] (under some nonessential restrictions) that

$$\|Mf\|_{L^p} \leqslant C\|M^{\#}f\|_{L^p} \quad \text{modulo constants, } 1 < p < \infty, \tag{4}$$

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where M is the usual Hardy-Littlewood maximal operator. However, (4) is not true for $p = \infty$ since BMO is not contained in L^∞. Therefore we are interested in spaces with $M^\# f$ in some Orlicz space "near" L^∞.

Let φ be a nonnegative, increasing, convex function on $\mathbf{R}^+(\varphi(0) = 0)$, and let φ^* be its convex conjugate. We define the Orlicz space L_φ as the space of functions f with norm (see [3])

$$\|f\|_{L_\varphi} = \inf_{\lambda>0} \frac{1}{\lambda}\left[\int_{\mathbf{R}^n} \varphi(\lambda|f(x)|)\,dx + 1\right] < \infty.$$

If φ satisfies

$$\varphi(2t) < C\varphi(t) \quad \text{for all } t > 0, \tag{5}$$

L_φ is separable with L_{φ^*} as its dual space.

Now we define $L_\varphi^\#$ as the space of functions f, modulo constants, such that $M^\# f \in L_\varphi$ with norm $\|f\|_{L_\varphi^\#} = \|M^\# f\|_{L_\varphi}$.

The proof of (4) in [1] works in following more general setting. If φ and φ^* satisfy (5) then (4) holds with the L^p-norms replaced by L_φ-norms, and then $L_\varphi^\# \equiv L_\varphi$.

We shall now define the Hardy space with Orlicz norm H_φ as the space of functions $f \in L_\varphi$ with the Riesz transforms $R_i f$, $i = 1, \ldots, n$, also in L_φ, and with the norm defined by

$$\|f\|_{H_\varphi} = \|f\|_{L_\varphi} + \sum_{i=1}^{n} \|R_i f\|_{L_\varphi}.$$

As for the H^p spaces in $\mathbf{R}^n$ (see [4]) there are several equivalent definitions.

If φ and φ^* satisfy (5) then $H_\varphi \equiv L_\varphi$.

We now come to the main result, which is a generalization of the result of Fefferman and Stein [1] that BMO is the dual space of H^1.

THEOREM. *If φ satisfies* (5) *then $L_{\varphi^*}^\#$ is the dual space of H_φ. More precisely, there is a dense subset H_φ^0 of H_φ such that* (i) *for every bounded linear functional l on H_φ there is a function $g \in L_{\varphi^*}^\#$ with $\|g\|_{L_{\varphi^*}^\#} < C\|l\|$ such that*

$$l(f) = \int_{\mathbf{R}^n} f(x)g(x)\,dx$$

for every $f \in H_\varphi^0$,

(ii) *if $g \in L_{\varphi^*}^\#$, then*

$$l_g(f) = \int_{\mathbf{R}^n} f(x)g(x)\,dx \quad \text{for } f \in H_\varphi^0$$

extends to a bounded linear functional on H_φ and $\|l_g\| \leqslant C\|g\|_{L_\varphi^\#}$.

The representation is unique, i.e. $I_g = 0$ if and only if g is constant.

For the proof we refer to [5]. The only interesting case of the Theorem is when φ does not satisfy (5), since otherwise it only says that $L^{\#}_{\varphi^*} \equiv L_{\varphi^*}$ is the dual space of $H_\varphi \equiv L_\varphi$.

REMARK. The method of the proof of the Theorem in [5] would work for more general spaces than Orlicz space with the norms defined in terms of distribution functions, for example Lorentz spaces.

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INSTITUT MITTAG-LEFFLER, AURAVÄGEN 17, S-182 62 DJURSHOLM, SWEDEN