THE CONJUGATING REPRESENTATION
OF A SEMISIMPLE ALGEBRAIC GROUP

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Let G be a connected, simply connected semisimple algebraic group over an algebraically closed field k of characteristic zero and let $A(G)$ denote the algebra of regular functions on G. We let G act morphically on G by conjugation. This defines a representation of G on $A(G)$. In analogy with the terminology for finite groups we call this the conjugating representation of G. Let $C(G)$ be the algebra of all (regular) class functions on G; equivalently $C(G) = A(G)^G$. Let Λ be a set of representatives for the equivalence classes of irreducible rational representations of G. If $\lambda: G \longrightarrow \mathrm{GL}(V_\lambda)$ is an element of Λ, let $d_\lambda = \dim V_\lambda$ and let $m_\lambda = \dim V_\lambda^T$, where T is a maximal torus of G. Let $A(G)_\lambda$ denote the isotopic component of $A(G)$ of type λ. Each $A(G)_\lambda$ is a finitely generated $C(G)$-module and $A(G)$ is the $C(G)$-module direct sum of the $A(G)_\lambda$, $\lambda \in \Lambda$.

THEOREM A. *Let $\lambda \in \Lambda$. Then there exists a G-stable vector subspace H_λ of $A(G)_\lambda$ such that the k-linear map $C(G) \otimes_k H_\lambda \longrightarrow A(G)$ defined by $c \otimes h \longrightarrow ch$ is an isomorphism of vector spaces. The rational G-module H_λ is equivalent to the direct sum of m_λ copies of V_λ. In particular, $A(G)_\lambda$ is a free C(G)-module of rank $d_\lambda m_\lambda$ and $A(G)$ is a free C(G)-module.*

For the adjoint action of G on the Lie algebra $\mathfrak{g}$ and the corresponding representation of G on the algebra $A(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}$, a similar theorem was proved by Kostant [1]. In the Lie algebra case, the proof of freeness is greatly simplified by the graded algebra structure of $A(\mathfrak{g})$. Our proof makes use of the results of Steinberg [4] (see also [3], [5]) on conjugacy classes in G and on the fibres of the morphism $\pi: G \longrightarrow k^r$, $r = \mathrm{rank}\ G$, given by the fundamental characters. We use the methods of commutative algebra. In particular, we require two separate applications of the Serre conjecture on projective modules over polynomial rings, which has recently been proved by Quillen [2].

If the base field k is of prime characteristic, we have a different proof that $A(G)$ is a free $C(G)$-module. In this case we have not been able to prove the existence of a G-stable vector subspace H of $A(G)$ such that the product map $C(G) \otimes_k H \longrightarrow A(G)$ is a vector space isomorphism.

In §1 we outline the proof of Theorem A. A detailed proof will appear elsewhere.

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1. Sketch of the proof of Theorem A. Roughly, there are three steps in the proof: (1) prove that $A(G)_\lambda$ is a projective $C(G)$-module; (2) apply the Serre conjecture to prove that $A(G)_\lambda$ is a free $C(G)$-module (Steinberg has shown that $C(G) \cong k[X_1, \ldots, X_r]$); (3) apply the Serre conjecture again to find a G-stable subspace H_λ as in Theorem A.

STEP (1). The projectivity of $A(G)_\lambda$ follows from Steinberg's results and 1.1 and 1.2 below.

1.1. Let C be a Noetherian integral domain which is a Jacobson ring and let M be a finitely generated C-module. Assume that there exists an integer q such that, for every maximal ideal m of C, the dimension of the (C/m)-vector space M/mM is equal to q. Then M is a projective C-module of rank q.

1.2. Let x be a regular element of G. Then, for every $\lambda \in \Lambda$, dim $V_\lambda^T =$ dim $V_\lambda^{Z G(x)}$.

STEP (2). The result below is due to Quillen [2]. The problem was posed almost twenty years ago by Serre and has become known as the Serre conjecture.

2.1. Let L be a principal ideal domain and let M be a finitely generated projective module over the polynomial ring $L[X_1, \ldots, X_r]$. Then M is a free module.

Since $C(G)$ is a polynomial ring over k and $A(G)_\lambda$ is a projective $C(G)$-module, it follows that $A(G)_\lambda$ is free.

STEP (3). Let $(f_1, \ldots, f_n)$ be a basis of the free $C(G)$-module $A(G)_\lambda$; here $n = d_\lambda m_\lambda$. The vector subspace of $A(G)_\lambda$ spanned by $f_1, \ldots, f_n$ is not necessarily G-stable. We wish to find a basis which spans a G-stable subspace. Let $g \in G$. Then $g \cdot f_j = \Sigma_{i=1}^n \alpha_{ij}(g) f_i$, where each $\alpha_{ij}(g)$ is an element of $C(G)$. This determines a homomorphism (of abstract groups) $\alpha \colon G \longrightarrow \mathrm{GL}_n(C(G))$, $\alpha(g) = (\alpha_{ij}(g))$. Identify $C(G)$ with the algebra of polynomial functions on k^r and define $\eta \colon G \times k^r \longrightarrow \mathrm{GL}_n(k)$ by $\eta(g, y) = (\alpha_{ij}(g)(y))$. With some work we can prove that η is a morphism of algebraic varieties. For each $y \in k^r$, define a rational representation $\eta_y \colon G \longrightarrow \mathrm{GL}_n(k)$ by $\eta_y(g) = \eta(g, y)$. Then $(\eta_y)_{y \in k^r}$ is a parametrized family of equivalent representations of G on k^n. Let $\mathcal{R}$ be the set of all rational representations of G on k^n which are equivalent to η_0. Then $\mathrm{GL}_n(k)$ acts transitively on $\mathcal{R}$ and the stabilizer, H, of η_0 is isomorphic to $\mathrm{GL}_{m_\lambda}(k)$. Thus η determines a map $\beta \colon k^r \longrightarrow \mathrm{GL}_n(k)/H$ and this map is a morphism. (The easiest way to see this is to consider the corresponding Lie algebra representations $d\eta_y \colon \mathfrak{g} \longrightarrow \mathfrak{gl}_n(k)$.) Now $\mathrm{GL}_n(k) \longrightarrow \mathrm{GL}_n(k)/H$ is a locally trivial principal H-bundle and H is isomorphic to $\mathrm{GL}_{m_\lambda}(k)$. Hence β determines a vector bundle of rank m_λ on k^r. Since vector bundles on k^r correspond to projective $C(G)$-modules, the Serre conjecture implies that this vector bundle is trivial. The triviality of this vector bundle allows us to find a new basis of the $C(g)$-module $A(G)_\lambda$ such that the vector subspace H_λ spanned by this basis is G-stable. This completes the outline of the proof.

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