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PERTURBATION AND ANALYTIC CONTINUATION OF GROUP REPRESENTATIONS

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ABSTRACT. I introduce a theory of noncommutative bounded perturbations of Lie algebras of unbounded operators. When applied to group representations, it leads to an analytic embedding of the dual object of some semisimple Lie groups into the bounded operators on corresponding Hilbert spaces of K-finite vectors.

1. **Introduction.** I announce a general theorem on analytic continuation of group representations which is based on perturbation theory for linear operators. This result is a contribution of the author to a series of joint results with R. T. Moore reported in detail in [3]. Applications of the theorem to quasi-simple Banach representations of $SL(2, \mathbf{R})$, due to Moore, will be announced separately by him. The theorem introduces a perturbation theory for representations of Lie groups which generalizes the classical perturbation theory (due to R. S. Phillips [2, p. 389]) for one-parameter (semi) groups of bounded linear operators on a Banach space. Let $\{\pi(t): -\infty < t < \infty\}$ be such a strongly continuous one-parameter group (C_0 group) acting on a Banach space E. Let A be the infinitesimal generator of π, and let U be a "small" (bounded, say) perturbation of A, $B = A + U$. Then B generates a C_0 group $\{\pi_U(t)\}$ on E, and this group depends analytically on U (in a sense which is specified in [2, p. 404]). In my theorem the real line $\mathbf{R}$ is replaced by a Lie group G, and A is replaced by a Lie algebra L of unbounded operators in E. U is going to be a tuple $(U_1, \ldots, U_r)$ of bounded operators. In that way I obtain a surprisingly simple analytic continuation picture for a wide class of induced representations, and other unitary and nonunitary representations.

2. **Assumptions.** I first restrict the class of perturbations U to be considered. In order to make sure that π_U is a representation of the same group for all U, I assume that the corresponding infinitesimal operator Lie algebras L_U are all algebraically isomorphic.

Let D be a linear space. Let $\mathfrak{A}(D)$ be the algebra of linear endomorphisms of D. It is also a real Lie algebra when equipped with the commutator bracket, $[A, B] = AB - BA$ for $A, B \in \mathfrak{A}(D)$. The Lie algebra L generated by a subset S of $\mathfrak{A}(D)$ is defined to be the smallest *real* Lie subalgebra of $\mathfrak{A}(D)$ which contains

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S. (The elements in L are real linear combinations of elements in S and commutators, possibly iterated, of such elements.)

Let L_0 be a finite-dimensional Lie subalgebra of $\mathfrak{A}(D)$, and let $A = (A_1, \cdots, A_r)$ be a basis for L_0. The order of the operators A_k is essential only when addition of operator tuples is performed: $A + U = B$ with $B_k = A_k + U_k$ and $U_k \in \mathfrak{A}(D)$ for $1 \leq k \leq r$. I consider r-tuples U with the property that the Lie algebra L_U generated by $A + U$ is algebraically isomorphic to some fixed finite-dimensional Lie algebra $\mathfrak{g}$ for all U: $L_U \approx \mathfrak{g}$. The set of such r-tuples is denoted by $\mathcal{U}$. Then there are simply connected Lie groups G (resp. G_0) with Lie algebras $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) such that $L_0 \approx \mathfrak{g}_0$.

I restrict the class $\mathcal{U}$ of perturbations further. Let $\|\cdot\|$ be a fixed norm on D. Put $\|x\|_0 = \|x\|$ and $\|x\|_n = \max\{\|A_{i_1} \cdots A_{i_n} x\|: 0 \leq i_k \leq r\}$ for $x \in D$ and $n = 1, 2, \ldots$. (Define A_0 to be the identity I on D.) An element $V \in \mathfrak{A}(D)$ is said to be $\|\cdot\|_n$-bounded if there is a finite constant c_n such that $\|Vx\|_n \leq c_n \|x\|_n$ for all $x \in D$. Let D_n be the completion of $(D, \|\cdot\|_n)$ for $n = 0, 1, \ldots$. Put $D_0 = E$. I assume that the operators A_k are closable when viewed as unbounded operators in E. Hence $D_1 \subset E$ (cf. [3]). If V is $\|\cdot\|_0$-bounded, it extends to a bounded operator on E, $\bar{V} \in L(E)$. Let $V \in \mathfrak{A}(D)$ be $\|\cdot\|_0$-bounded. Then V is $\|\cdot\|_n$-bounded for given n iff the commutators $[A_{i_1}, [A_{i_2}, \ldots, [A_{i_n}, V] \ldots]]$ are $\|\cdot\|_0$-bounded for all i $(0 \leq i_k \leq r)$.

Consider the following subset $\mathcal{V}$ of $\mathcal{U}$: $U = (U_1, \ldots, U_r)$ belongs to $\mathcal{V}$ if and only if each U_k is $\|\cdot\|_0$-bounded and one of the following two conditions is satisfied:

(i) each U_k is $\|\cdot\|_n$-bounded for all n; or

(ii) $\{A_k + U_k\}$ is a basis for L_U and each U_k is $\|\cdot\|_1$-bounded.

3. THE THEOREM. *Let $L_0 \subset \mathfrak{A}(D)$ be an operator Lie algebra, $A = (A_1, \cdots, A_r)$ a basis for L_0, and let the class $\mathcal{V}$ of bounded perturbations be as described above. Suppose that L_0 exponentiates to a C_0 representation π of G_0 on E.*

(a) *Then L_U exponentiates to a C_0 representation of G on E for all $U \in \mathcal{V}$. We denote the exponential by π_U.*

(b) *Let Ω be a complex domain (in one or several dimensions). Let $z \longrightarrow U(z) = (U_1(z), \ldots, U_r(z))$ be an analytic function which is defined on Ω and has its range in $\mathcal{V}$. Then $\pi_{U(z)}$ is analytic as a function of z, i.e., $z \longrightarrow \pi_{U(z)}(g)$ is analytic for all $g \in G$.*

(c) *The representations π and π_U have the same space of C^∞-vectors for all $U \in \mathcal{V}$.*

REMARK. A suitable class of analytic perturbations U gives representations π_U which have the same space of analytic vectors as π.

The proof is based on two exponentiation theorems due to the co-authors of

[3]. I state those theorems as lemmas here. They are significant improvements of results announced in [4], and appear below for the first time in their strengthened form.

LEMMA 1. *Let D be a normed linear space, and E the corresponding completion. Let $\mathsf{L} \subset \mathfrak{A}(D)$ be a finite-dimensional Lie algebra. Suppose L is generated (in the Lie sense) by a subset S such that every $A \in \mathsf{S}$ is closable and the closure $\bar{A}$ generates a C_0 group $\{\pi(t, A): t \in \mathbf{R}\} \subset L(E)$.*

If D is invariant under $\pi(t, A)$ for $t \in \mathbf{R}$ and $A \in \mathsf{S}$, and $t \longrightarrow \|B\pi(t, A)x\|$ is locally bounded for all $B, A \in \mathsf{S}$ and $x \in D$, then L exponentiates.

LEMMA 2. *Let L and S be as above. (This means that we have C_0 groups $\{\pi(t, A)\}$ for $A \in \mathsf{S}$, and there are finite constants ω_A such that*

$$\sup_t e^{-|t|\omega_A} \|\pi(t, A)\| < \infty.)$$

Let $B_1, \ldots, B_d$ be a basis for L. Put $B_0 = I$, and

$$\|x\|_1 = \max\{\|B_i x\|: 0 \leqslant i \leqslant d\}$$

for $x \in D$.

Suppose each $A \in \mathsf{S}$ satisfies the condition: **(GD)** There are complex numbers $\lambda_\pm$ *such that* Re $\lambda_+ > \omega_A + |\mathrm{ad}\, A|$, Re $\lambda_- < -\omega_A - |\mathrm{ad}\, A|$, *and the ranges of* $\lambda_\pm I - A$ are $\|\cdot\|_1$*-dense in D. Then L exponentiates.*

PROOF SKETCH (a). Suppose L_0 exponentiates to a representation π. Let $U \in \mathsf{V}$, and suppose that (i) holds. Then one may apply bounded Phillips perturbations to each of the spaces $D_n = D_n(\pi)$ (cf. [1, Proposition 1.1]) and conclude that each $\bar{B}_k = \bar{A}_k + \bar{U}_k$ generates a C_0 group $\pi(t, B_k)$ which leaves D_∞ invariant. So Lemma 1 applies to L_U, with D replaced by D_∞.

If (ii) holds, then apply Lemma 2 to L_U. Bounded Phillips perturbation in D_1 shows that $\pi(t, B_k)$ restricts to a C_0 group in $L(D_1)$. Condition (GD) is a simple consequence of this.

REMARK. The lemmas are hard to apply *directly* to operator Lie algebras that arise in applications. Fortunately many of these can be shown to be perturbations of a base-point Lie algebra to which the lemmas easily apply.

At this point I have verified, using the theorem, that the dual $\hat{G}$ of the 3- or the 15-dimensional conformal group is analytically embedded via $\pi_U \longrightarrow U$ in $B(H)$ for a common Hilbert space H. The range consists of operators which are linear combinations of bounded shifts modulo the compacts (and occasionally Hilbert-Schmidts). This gives new and simple metrics on $\hat{G}$, and thus realizes ideas that were recently suggested to me by Professor I. E. Segal.

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