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THE HOMEOMORPHISM GROUP OF A COMPACT Q-MANIFOLD IS AN ANR

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In this paper we will let $Q = \Pi_{i=1}^{\infty} [-1, 1]$ denote the Hilbert cube, $l_2 = \{(x_i) | \Sigma |x_i|^2 < \infty\}$ denote separable Hilbert space, $C(X, X)$ denote the continuous functions from X to X with the sup norm topology with respect to a suitable metric on X, $H(X) \subset C(X, X)$ denote the self-homeomorphisms of X, and $\bar{H}(X)$ denote the closure of $H(X)$ in $C(X, X)$. By a manifold, we will mean a separable metric space modelled on $\mathbf{R}^n$, Q, or l_2. The following question is of interest to both infinite dimensional and finite dimensional topologists. Compare [**A–B**, p. 792].

Q1. If M is a manifold modelled on $\mathbf{R}^n$ or Q, is $H(M)$ an l_2-manifold?

General results which are known include:

(1) Geoghegan [G] has shown that $H_A(M) \times l_2 \overset{\text{homeo}}{\approx} H_A(M)$. Here, $H_A(M)$ is the space of homeomorphisms of M which fix a proper closed subset A of M.

(2) Toruncyzk [T] has shown that the product of l_2 with a complete separable metric ANR is an l_2-manifold.

These results reduce Q1 to the question:

Q1$'$. If M is a compact manifold, is $H(M)$ an ANR?

(3) Mason and Luke and Mason [M], [L-M] have answered Q1$'$ affirmatively when M is a two-dimensional manifold.

Using completely different techniques, we have proven the analogous result for Q-manifolds. Recall that a closed set $A \subset M$ is called a Z-set if there are maps of M into $M - A$ which are arbitrarily close to the identity.

THEOREM 1. *If M is a compact Q-manifold and $A \subset M$ is a Z-set, then $H_A(M)$ is an ANR.*

We require more notation in order to state our next result. Let X and Y be metric spaces, and let β be an open cover of Y. A homotopy $H: X \times I \longrightarrow Y$ is called a *β-homotopy* if for each $x \in X$ there is an element $U_x \in \beta$ such that

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$H(\{x\} \times I) \subset U_x$. A map $f: X \longrightarrow Y$ will be called a *β-equivalence* if there is a map $g: Y \longrightarrow X$ such that $f \circ g$ is β-homotopic to the identity and $g \circ f$ is $f^{-1}(\beta)$-homotopic to the identity. f is called a *fine homotopy equivalence* if f is a β-equivalence for each open cover β of Y.

Basic work on fine homotopy equivalences has been done by Lacher, Price, Kozlowski, and Haver, among others.

If f, g: $X \longrightarrow Y$ are maps and β is an open cover of Y, f and g are said to be β-close if for each $x \in X$ there is an open $U_x \in \beta$ such that $f(x), g(x) \in U_x$. A map $f: X \longrightarrow Y$ is called a *near-homeomorphism* if for each open cover β of Y there is a homeomorphism $h: X \longrightarrow Y$ which is β-close to f. If M is compact, then $f: M \longrightarrow M$ is in $\overline{H(M)}$ if and only if f is a near-homeomorphism.

Armentrout **[A]**, Siebenmann **[S]**, and Chapman **[$\mathbf{Ch_1}$]** have proven (roughly) that the near-homeomorphisms are the fine homotopy equivalences between n-manifolds of dimensions $n \leqslant 3$, $n \geqslant 5$, and for Q-manifolds, respectively. For exact statements of the theorems, consult the references cited above.

Our main tool in the proof of Theorem 1 is a parametrized estimated version of Chapman's theorem. We state the corresponding unparametrized result as Theorem 2. Recall that a map $f: X \longrightarrow Y$ is *proper* if $f^{-1}(K)$ is compact for each compact $K \subset Y$. A β-equivalence f is proper if f, g and the homotopies are proper.

THEOREM 2. *If M is a Q-manifold and α is an open cover of M, then there is an open cover β of M such that if N is a Q-manifold and $f: N \longrightarrow M$ is a proper β-equivalence then f is α-close to a homeomorphism.*

Note that β depends on M and α but not on N. As an immediate corollary, we obtain

THEOREM 2′. *If K is a countable locally finite polyhedron, then there is an open cover β of K such that if $f: L \longrightarrow K$ is a proper β-equivalence then f is a simple homotopy equivalence.*

This is a considerable generalization of the theorem that CE maps between locally finite polyhedra are simple homotopy equivalences **[$\mathbf{Ch_2}$]**.

The techniques of our proof yield an even stronger result for l_2-manifolds.

THEOREM 3. *If M and N are l_2-manifolds, α is an open cover of M, and $f: N \longrightarrow M$ is an α-equivalence, then f is Stα-close to a homeomorphism.*

Thus, in the category of l_2-manifolds, the near-homeomorphisms are precisely the fine homotopy equivalences.

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