EXISTENCE THEOREMS ACROSS A POINT OF RESONANCE

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1. Let us consider the equation

$$(1) \qquad\qquad Ex + \alpha x = Nx,$$

where $E: D(E) \longrightarrow S$, $D(E) \subset S$, is a linear not necessarily bounded operator in a real Hilbert space S with $1 \leqslant \dim W < \infty$, $W = \ker E$, and $N: S \longrightarrow S$ is a continuous nonlinear operator. In terms of the alternative method (see Cesari [1], [2]), let $P: S \longrightarrow S$ be the orthogonal projector with range $PS = S_0 = W = \ker E$, let $S_1 = (I - P)S$, and let S_1 be also the range of E. Let $H: S_1 \longrightarrow S_1$ denote a linear, bounded, compact operator (a partial inverse of E) such that the usual relations of the alternative method (selfadjoint case) hold: (h_1) $H(I - P)E = I - P$; (h_2) $EP = PE$; (h_3) $EH(I - P) = I - P$. Let $(\ ,\ )$ and $\|\ \|$ denote inner product and norm in S, and let $L = \|H(I - P)\|$.

I (AN EXISTENCE THEOREM "AT RESONANCE"). *Under the above hypotheses, let us assume that*

(B) *there is a constant $J_0 > 0$ such that $\|Nx\| \leqslant J_0$ for all $x \in S$; and*

(N_0) *there is a constant $R_0 \geqslant 0$ such that $(Nx, x^*) \leqslant 0$ [or always $(Nx, x^*) \geqslant 0$] for all $x \in S$, $x^* \in S_0$ with $Px = x^*$, $\|x^*\| \geqslant R_0$, $\|x - Px\| \leqslant LJ_0$. Then the equation $Ex = Nx$ has at least a solution $x \in X$.*

We refer to Cesari and Kannan [6] for a proof of I by Schauder's fixed point theorem, and to Kannan and McKenna [8] for a proof based on a Leray-Schauder argument. It has been pointed out that the hypotheses contained in recent specific and relevant theorems by Landesman and Lazer [9], Williams [13], Nečas [11], and Lazer and Leach [10] concerning existence at resonance for boundary value problems for ordinary and partial differential equations, imply conditions (B) and (N_0). But the same hypotheses of the aformentioned specific theorems also imply the stronger condition (N_ϵ) below. In turn, this condition has a stronger implication which we present here.

II (AN EXISTENCE THEOREM "ACROSS A POINT OF RESONANCE"). *Under the same general hypotheses above, let us assume that* (B) *holds, and that*

(N_ϵ) *there are constants $R_0 \geqslant 0$, $\epsilon > 0$, $K > LJ_0$ such that $(Nx, x^*) \leqslant$*
$-\epsilon\|x^*\|$ (or always $(Nx, x^*) \geqslant \epsilon\|x^*\|$) for all $x \in S$, $x^* \in S_0$ with $Px = x^*$, $\|x^*\|$
$\geqslant R_0$, $\|x - Px\| \leqslant K$.

Then, there are constants $\alpha_0 > 0$, $C > 0$ such that, for every $|\alpha| \leqslant \alpha_0$, the
equation $Ex + \alpha x = Nx$ has at least a solution x with $\|x\| \leqslant C$.

We refer to Cesari [3] for a proof of II by Schauder's fixed point theorem.
Note that, in equation (1), the parameter α can "move across the point of reso-
nance $\alpha = 0$", without the solution becoming infinite, due to the nonlinearity of
the problem. Our result II, in the large and under sole qualitative hypotheses,
seems to be new and of some physical significance. In [3], [4], [5] Cesari has
also proved more general statements, with S replaced by Banach spaces X, Y, for
$N: X \longrightarrow Y$ continuous and of slow growth, and the term αx replaced by αAx,
with $A: X \longrightarrow Y$ any continuous, nonlinear operator, mapping bounded sets into
bounded sets. As applications of II we restate, for instance, the theorems of
Landesman and Lazer, Williams, and Lazer and Leach, in a stronger form.

2. Let us consider the ordinary differential equation

(2) $x'' + (m^2 + \alpha)x + h(x) = p(t)$, x scalar,

where m is an integer, h is a real valued function in $(-\infty, +\infty)$, $p(t)$ is a 2π-
periodic function, and α is a real parameter.

   III. *If p is continuous and 2π-periodic, if h is continuous with $|h(s)| \leqslant M$*
for all real s, and if there are constants $c < d$, $C < D$ such that $h(s) \leqslant C$ for
$s \leqslant c$; $h(s) \geqslant D$ for $s \geqslant d$, and moreover $(A^2 + B^2)^{1/2} < 2(D - C)$, where $A =$
$\int_0^{2\pi} p(t)\cos mt\, dt$, $B = \int_0^{2\pi} p(t)\sin mt\, dt$, then there are constants $\alpha_0 > 0, M > 0$
such that for every $|\alpha| \leqslant \alpha_0$, equation (2) has at least a 2π-periodic solution $x(t)$,
$-\infty < t < +\infty$, with $|x(t)| \leqslant M$.

   For $\alpha = 0$ this theorem was proved by Lazer and Leach [8]. The same
statement holds also for h replaced by $-h$.

3. Let us consider the partial differential equation

(3) $Ex + \alpha x + h(x) = f(t)$, $t \in G$,

where G is a bounded domain in R^ν of points $t = (t_1, \ldots, t_\nu)$, E is the differ-
ential operator of order $2m$ in G defined by

$$Ex = \sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\beta|} D^\beta(a_{\alpha\beta}(t)D^\alpha x), \quad t \in G,$$

with $a_{\alpha\beta} = a_{\beta\alpha}$, and $\Sigma a_{\alpha\beta}(t)\zeta^\alpha \zeta^\beta \geqslant c|\zeta|^{2m}$ for some constant $c > 0$ and where
Σ ranges over all $|\alpha| = |\beta| = m$. In order to simplify the exposition, and follow-

ing de Figueiredo, we limit ourselves to strong solutions, and therefore under some smoothness hypotheses, say $a_{\alpha\beta} \in C^{|\beta|}$ in $\overline{G}$, and ∂G of class C^{2m}. We assume that the homogeneous linear Dirichlet problem $Ex = 0, x \in H_0^m \cap H^{2m}$, has nontrivial solutions, thus, $1 \leq \dim W < \infty$, $W = \ker E$. Let $h(s), -\infty < s < +\infty$, be a continuous real valued function, for which the limits $R = h(+\infty)$ and $r = h(-\infty)$ exist and are finite. For every element $w \in W$ let G^+, G^- denote the subsets of all $t \in G$, where $w(t) \geq 0$ and $w(t) \leq 0$, respectively, and let $w^+ = \int_{G^+} |w| \, dt$, $w^- = \int_{G^-} |w| \, dt$.

IV. *Under the conditions above and for $f \in L_2(G)$, if for every $w \in W$ we also have $Rw^- - rw^+ < \int_G f(t)w(t) \, dt < Rw^+ - rw^-$, then there are constants $\alpha_0 > 0, C > 0$ such that, for every real α with $|\alpha| \leq \alpha_0$, equation (3) has at least a solution $x \in H_0^m \cap H^{2m}$ with $\|x\|_{2m} \leq C$.*

The same statement holds also for h replaced by $-h$. For $\alpha = 0, m = 1$ this statement was proved by Landesman and Lazer [9], and for $\alpha = 0, m \geq 1$ by Williams [13]. The case in which h depends on t, x, and the partial derivatives of x of orders $\leq 2m - 1$ has been studied by de Figueiredo [7] for $\alpha = 0$, and his statement too can be restated in the strong form above. Analogous results have been proved for the case in which h may grow with x of order r, $0 \leq r < 1$, by de Figueiredo [7] for $\alpha = 0$, and these statements too can be given the strong form above in terms of the alternative method (cf. Cesari [4]). For nonselfadjoint problems, see Shaw [12] and Cesari [4], [5].

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