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DUALITY AND KATO'S THEOREM ON SMALL PERTURBATIONS

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ABSTRACT. The main result is a theorem on stability of index under small perturbations in locally convex spaces, which reduces for Banach spaces to the familiar theorem of T. Kato.

It is a well-known result of Gohberg and Kreĭn [2] and Kato [3] that if T is a semi-Fredholm operator and P a bounded operator of norm small enough, then $T + P$ is a semi-Fredholm operator with the same index as T. Kato gives a precise upper bound of the norm of P: $\|P\| < \gamma(T)$, where $\gamma(T)$ essentially is $\|\hat{T}^{-1}\|^{-1}$, $\hat{T}$ being the one-to-one operator induced by T. In geometric terms, this may be expressed as $TB \supset \lambda B' \cap R(T)$, $PB \subset \mu B'$ and $0 \leqslant \mu < \lambda$, B, B' being the unit balls of E, F.

Some results concerning small bounded perturbations of Φ_--operators in more general locally convex spaces are given in [7], [4], but they do not fully render the precise Kato theorem in case of Banach spaces.

By using Kato's theorem in the dual, we obtain some results (Propositions 1 and 3) which do constitute an extension of Kato's theorem on small perturbations of Φ_--operators, and refine several results in [7], [4].

In the sequel, E, F always denote two Hausdorff locally convex spaces, and T, P two (linear) operators from E into F such that $[D(T)]^- \subset D(P)$, $[D(T)]^-$ being the closure of the domain of T. Let $N(T)$ and $R(T)$ denote the kernel and the range of T. By neighborhood we mean an absolutely convex neighborhood of the origin. A *disk* is an absolutely convex set.

The operator T is *open* (resp. *almost open*) if TU (resp. $[TU]^-$) is a neighborhood in $R(T)$, for any neighborhood $U \subset E$. T is a Φ_- (resp. Φ_+)-*operator* if T is open, has a closed graph (in $E \times F$) and a closed range, and codim $R(T) < \infty$ (resp. dim $N(T) < \infty$). The *index* of T is then defined as $\mathrm{ind}(T) = \dim N(T) - \operatorname{codim} R(T)$ (we do not distinguish between different cardinalities of infinity).

PROPOSITION 1. *Let T be an almost open operator with* codim$[R(T)]^-$ $< \infty$, *and P a continuous operator.*

(1) *Assume that there exists a base of neighborhoods U in E such that*

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$PU \cap [R(T)]^- \subset \epsilon[TU]^-$, $0 < \epsilon < 1$.

(2) *Assume further that for some neighborhood* $U_0 \subset E$, *there are bounded disks* B, B' *such that* $PU_0 \subset [TB]^- + B'$.

Then $T + P$ *is almost open and* $\operatorname{codim}[R(T + P)]^- \leqslant \operatorname{codim}[R(T)]^-$. *If* E *is a Fréchet space, or, more generally, fully complete* [6], *and* T *has a closed graph, then* T *and* $T + P$ *are* Φ_-*-operators and* $\operatorname{ind}(T + P) = \operatorname{ind}(T)$.

COROLLARY 2. *Conditions* (1) *and* (2) *are satisfied, and Proposition* 1 *holds true if there exist a neighborhood* U_0, *a bounded disk* B *and* $0 < \epsilon < 1$ *such that* $B \subset \epsilon U_0$, $PU_0 \cap [R(T)]^- \subset [TB]^-$ *and* PU_0 *is bounded.*

The proof of Proposition 1 uses duality and Kato's theorem applied to the Banach spaces generated by closed equicontinuous sets, and is very much similar to those in [4]. Some technical modifications lead to the following more general formulation:

PROPOSITION 3. *Let* T *be an almost open operator with* $\operatorname{codim}[R(T)]^- < \infty$ *and* P *a continuous operator.*

(1′) *Assume that there is a base of neighborhoods* $\mathfrak{U}$ *in* E *such that for any* $U \in \mathfrak{U}$, *there exist a finite-dimensional subspace* $N \subset [R(T)]^-$, *and* $0 < \epsilon < 1$, *for which* $PU \cap [R(T)]^- \subset \epsilon[TU]^- + N$.

(2′) *Assume further that for some neighborhood* $U_0 \subset E$, *a finite-dimensional subspace* N_0, *and bounded disks* B, B', $PU_0 \subset [TB]^- + B' + N_0$.

Then $T + P$ *is almost open and*

$$\operatorname{codim}[R(T + P)]^- \leqslant \operatorname{codim}[R(T)]^- + \dim(N + N_0) < \infty.$$

If E *is fully complete, and* T *has a closed graph, then* $T + P$ *and* T *are* Φ_-*-operators and* $\operatorname{ind}(T + P) = \operatorname{ind}(T)$.

COROLLARY 4. *Condition* (1′) *is satisfied if there exist a neighborhood* $U \subset E$, *a bounded disk* B_0, *a precompact disk* K, *a finite-dimensional subspace* N' *and* $0 < \epsilon < 1$ *such that* $B_0 \subset \epsilon U$ *and* $PU \cap [R(T)]^- \subset [TB_0]^- + K + N'$.

Both conditions (1′) *and* (2′) *are satisfied in particular if* $PU \subset [TB_0]^- + K + N'$.

If $PU \subset [TB_0]^- + N'$ *then* $[R(T + P)]^- + N' = [R(T)]^- + N'$.

REMARKS. Again, by application of a result of Kato in the dual, it could be shown that $\operatorname{codim}[R(T + \lambda P)]^-$ is constant for $|\lambda| \neq 0$ and small enough.

Corollary 4 yields at the same time Theorem 4.b, and the remarks following Theorems 2 and 4 in [7], where the perturbations are of the type $PU \subset [TB_0]^- + N'$ and $PU \subset [TB_0]^- + K$ (K compact). It also provides another short proof of the main part of Theorem 2 in [8] (see also [5]), and shows that precompact perturbations of Φ_--operators may be reduced to small perturbations.

We would like also to point out that duality is a convenient tool to study the stability of "almost-openness" of Φ_+ and Φ_--operators under small or precompact perturbations. The stability of the index is readily obtained when suitable assumptions of completeness are placed on the spaces in such a way that the perturbed operator becomes a Φ_+ or Φ_--operator.

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