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ON MONOTONE VS. NONMONOTONE INDUCTION

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1. Introduction. For definitions and notation in what follows, see [4] and [5]. If A is an infinite set and $\varphi(y_1 \cdots y_n, R, Y_1 \cdots Y_m) = \varphi(\overline{y}, R, \overline{Y})$ is a second order relation on A, we call φ *operative* if R is n-ary. For such a φ let

$$I_\varphi^\xi = \bigcup_{\eta<\xi} I_\varphi^\eta U\left\{(\overline{y}, \overline{Y}): \varphi\left(\overline{y}, \left\{\overline{y}: (\overline{y}, \overline{Y}) \in \bigcup_{\eta<\xi} I_\varphi^\eta\right\}, \overline{Y}\right)\right\} \quad \text{and} \quad I_\varphi = \bigcup_\xi I_\varphi^\xi.$$

If $\mathcal{F}$ is a collection of second order relations (for simplicity *collection of operators*) on A, then $\mathcal{F}$-IND^2 is the class of all second order relations of the form $\psi(\overline{x}, \overline{Y}) \Leftrightarrow I_\varphi(\overline{a}, \overline{x}, \overline{Y})$, for some operative $\varphi(\overline{u}, \overline{x}, R, \overline{Y})$ in $\mathcal{F}$ and constants $\overline{a}$ from A. As in [5] $\mathcal{F}$-IND is the class of all *relations* on A which are in $\mathcal{F}$-IND^2. We let $\mathcal{F}^{\text{mon}}$ be the collection of all operative $\varphi(\overline{y}, R, \overline{Y})$ in $\mathcal{F}$ which are *monotone* on R and we put $\neg\mathcal{F} = \{\neg\varphi: \varphi \in \mathcal{F}\}$. A collection of operators $\mathcal{F}$ on A is *adequate* if it contains all the $\Pi_1^0(\mathcal{C})$ second order relations, where $\mathcal{C}$ is a coding scheme on A and is closed under $\wedge, \vee, \exists^A$ and trivial combinatorial substitutions. Let $WF(S) \Leftrightarrow S$ be a well-founded relation on $A \Leftrightarrow \neg \exists a_0 a_1 a_2 \cdots \forall i(a_{i+1}, a_i) \in S$.

THEOREM 1. *Let $\mathcal{F}$ be an adequate collection of operators on an infinite set A. If $WF \in \neg\mathcal{F}$ and $\neg\mathcal{F} \subseteq \mathcal{F}^{\text{mon}}$-IND^2, then $\mathcal{F}$-IND^2 = $\mathcal{F}^{\text{mon}}$-IND^2.*

2. Elementary induction. Let $\mathcal{EL}$ be the collection of all the elementary second order relations on a structure $\mathcal{A} = \langle A, R_1 \ldots R_l\rangle$ and let $\mathcal{EL}^+$ be the subcollection of $\mathcal{EL}^{\text{mon}}$ consisting of all operative $\varphi(\overline{x}, R, \overline{Y})$ which are definable by positive in R elementary formulas. One usually writes $\mathcal{EL}^+$-IND^2 = IND^2 and $\mathcal{EL}^+$-IND = IND. Clearly IND$^2 \subseteq \mathcal{EL}^{\text{mon}}$-IND$^2 \subset \mathcal{EL}$-IND^2 and it is well known that IND^2 is a tiny part of $\mathcal{EL}$-IND^2 for (say) almost acceptable $\mathcal{A}$'s. By a basic result of Kleene and Spector for ω and Barwise-Gandy-Moschovakis in general (see [4, §8A]), on every *countable* almost acceptable structure, IND^2 = $\mathcal{EL}^{\text{mon}}$-IND^2 (= Π_1^1). On the other hand, letting $WF^n(S) \Leftrightarrow S$ is a 2n-ary relation on A which is well founded (viewed as binary on A^n), we have

COROLLARY 1. *Let $\mathcal{A}$ be an infinite structure such that each WF^n is elementary. Then $\mathcal{EL}^{\text{mon}}$-IND^2 = $\mathcal{EL}$-IND^2.*

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A more detailed level-by-level version of Corollary 1 is the following, where we just write Σ^0_m, Π^0_m instead of $\Sigma^0_m(\mathcal{C})$, $\Pi^0_m(\mathcal{C})$, where $\mathcal{C}$ is a hyperelementary coding scheme on $\mathfrak{A}$.

COROLLARY 2. *Let $\mathfrak{A}$ be an almost acceptable structure. If $m \geqslant 2$ and $WF \in \Pi^0_m$, then for all $n \geqslant m$, $\Sigma^0_n\text{-IND}^2 = (\Sigma^0_n)^{\text{mon}}\text{-IND}^2$.*

So, for example, in the structure of analysis **R** this says that Σ^1_n monotone operators on R inductively define the same relations as arbitrary Σ^1_n operators, when $n \geqslant 2$. Similarly for Σ^1_n. The following rather curious result can be also established by the methods used to prove Theorem 1. If $\mathfrak{A} = \langle A, R_1 \cdots R_l\rangle$ is a structure, by an *elementary quantifier Q* on A we understand a quantifier on A which viewed as a second-order relation is elementary.

THEOREM 2. *Let $\mathfrak{A}$ be an acceptable structure in which WF is elementary. There is an elementary quantifier Q on A such that for every inductive relation R on A, there is an inductive relation R^* on A such that $\neg R(\bar{x}) \Leftrightarrow QyR^*(\bar{x}, y)$.*

This should be compared with a result of Moschovakis [3] in higher type recursion, where "inductive" is replaced by "semirecursive in a total object of type $\geqslant 3$" and Q becomes the existential quantifier (on an appropriate space).

REMARKS. (i) We conjecture that in Theorem 1 (and correspondingly in Corollary 1) the hypothesis $WF \in \neg\, \mathcal{F}$ can be weakened to $WF \in \neg(\mathcal{F}^{\text{mon}}\text{-IND}^2)$. (ii) In a direction opposite to that of Corollary 1 one has the following theorem of Nyberg (unpublished): Let $\mathfrak{A}$ be almost acceptable. If $\text{IND} \not\subseteq -(\mathcal{EL}^{\text{mon}}\text{-IND})$, then $\mathcal{EL}^{\text{mon}}\text{-IND} = \text{IND}$. Thus for most structures occuring in practice, $\mathcal{EL}^{\text{mon}}\text{-IND}$ is either IND or $\mathcal{EL}\text{-IND}$.

3. Further corollaries and applications to Spector classes. An immediate consequence of Theorem 1 is also the following result of Harrington and Moschovakis [2]. (Given a structure $\mathfrak{A}$ and a quantifier Q on A we abbreviate by Q-IND the class of second order relations which are positive $\mathcal{L}^A(Q)$-inductive (see [4, p. 49]).

COROLLARY 3. (*Harrington-Moschovakis* [2]). *Let $\mathfrak{A}$ be an almost acceptable structure and let Q be a quantifier on A. If $\mathcal{F} = \neg\,(Q\text{-IND}^2)$, then $\mathcal{F}\text{-IND}^2 = \mathcal{F}^{\text{mon}}\text{-IND}^2$.*

This generalizes a result of Grilliot to the effect that over ω, $\Sigma^1_1\text{-IND}^2 = (\Sigma^1_1)^{\text{mon}}\text{-IND}^2$. The original proof of Corollary 2 in [2] yields the stronger statement that for $\mathcal{F} = \neg\,(Q\text{-IND}^2)$, $\mathcal{F}\text{-IND}^2 = \mathcal{F}^{\text{pos}}\text{-IND}^2$ and also shows that $\mathcal{F}\text{-IND}^2 = Q^+\text{-IND}^2$, where Q^+ is the *next quantifier* of Q (see [1]). Turning now to Spector classes we can obtain the following, where the notions involved are explained in [5].

THEOREM 3. *Let Γ be a Spector class on A, and let $\mathcal{F}$ be a reasonable,* *nonmonotone class of operators on A closed under* $\exists^A$. *If* $WF \in \neg \mathcal{F}$, *then* Γ *is* $\mathcal{F}$*-compact iff* Γ *is* $\mathcal{F}_*^{\mathrm{mon}}$*-compact, where* $\mathcal{F}_*^{\mathrm{mon}} = \{\varphi(R)\colon \varphi \in \mathcal{F}, \varphi \text{ monotone}\}$. *In particular if* $\mathcal{F}$ *is typical, nonmonotone,* $\mathcal{F}^{\mathrm{mon}}$-IND *is a Spector class iff* $\mathcal{F}^{\mathrm{mon}}$-IND $= \mathcal{F}$-IND.

Further applications of the methods developed here to the theory of "second order" Spector classes as well as details and proofs of the results announced here will appear elsewhere.

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