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Comparison theorems in Riemannian geometry, by Jeff Cheeger and David G. Ebin, North-Holland Mathematical Library, vol. 9, North-Holland, Amsterdam; American Elsevier, New York, 1975, viii + 174 pp., $20.95.

Differential geometry is an almost unique area within mathematics, since it involves both the old and the new in an essential way. Riemannian geometry itself has, of course, been around for over one hundred years: About twenty-five years ago geometers began to ask how the local curvature of Riemannian manifolds could influence their global properties. (There were clues that this was an interesting question, e.g., the theorem of Hadamard and Cartan that a complete simply connected Riemannian manifold of nonpositive sectional curvature was diffeomorphic to Euclidean space.) The major opening salvo in this campaign was Rauch's work, published in 1951, showing that a (positive definite) Riemannian manifold whose sectional curvature function is sufficiently close to the curvature of the usual metric on the sphere is, in fact, homeomorphic to the sphere. Rauch combined techniques whose roots lie in the classical work: Sturm-type theorems for the systems of linear ordinary differential equations which result from linearization of the geodesic equations, and the distance minimizing property of the geodesics. Berger, Klingenberg and Toponogov then developed the conditions on the curvature which assure that the manifold is homeomorphic to the sphere and analyzed what happens at the precise point that the conditions are violated. They also developed a refined and powerful methodology to deal with this type of problem. In the sixties the methods were successfully applied to two general problems: Find Rauch-type conditions on the curvature which would assure that the manifold is diffeomorphic to the sphere, and study general global properties of Riemannian manifolds whose sectional curvature has a fixed sign.

This superb book gives us a masterful and definitive account of this work. The authors assume that the reader knows very well the material in a solid year-long course in differential and Riemannian geometry. (Alas, this is a course that very few can be expected to have had!) After a short review of this foundational material–and their definition of this includes everything up to and including the Rauch Comparison Theorem–they give efficient and polished proofs of the major results within 167 pages!

Here are some highlights. Chapter 2 starts the real work with Toponogov's Theorem. This is a global inequality version of a relation between curvature, angles and lengths of sides of triangles, which–in an infinitesimal form–goes back to Riemann. Chapter 3 describes the curvature of homogeneous Riemannian spaces. Chapter 4 describes ideas and results of Morse Theory, which are to be used later. Chapters 5 and 6 present the basic results concerning a compact, simply connected Riemannian manifold M whose sectional curvature is positive and "pinched", i.e., lies in a fixed interval of the real numbers. If this interval lies (after normalization of the metric) in the half-open interval $(\frac{1}{4}, 1]$, then M is homeomorphic to the sphere. If the interval is $[\frac{1}{4}, 1]$, then M is either homeomorphic to or a sphere *or* the metric is isometric to a symmetric space. (For example, the projective spaces have–in the standard metric–a curvature which is pinched between $\frac{1}{4}$ and 1.) Chapter 7 deals with conditions on the curvature which assume that the manifold is diffeomorphic to the sphere. Chapter 8 presents more recent work–due mainly to Cheeger, Gromoll and Meyer–concerning a complete noncompact Riemannian manifold of M nonnegative curvature. The main result is the existence of a compact, totally convex, submanifold S such that M is diffeomorphic to the normal bundle of S. A typical example is the cylinder: S is then the horizontal circle. If the sectional curvature of M is strictly positive, then S reduces to a point. Finally, Chapter 9 presents some recent results on discrete groups of isometries of Riemannian manifolds of nonpositive curvature which can be proved using the direct geometric technique of the book.

With this book in hand to give perspective, we can see that the development of modern differentiable manifold ideas has been successfully and compatibly combined with classical geometric intuition and methodology. Most of the results do not require–as they do in many of the other avantegarde areas of mathematics, e.g., algebraic geometry–an elaborate formalism, but they do involve refreshingly old-fashioned geometric ingenuity. This area is one of the most attractive, interesting and important in modern geometry.

The expositional efficiency of the book has been achieved at the expense of the elimination of most references to the broader vistas of differential geometry. As a peripheral example, the local arc-length minimizing property of geodesics is proved using a result that is labelled only as Gauss' Lemma. One must be familiar with this material from another perspective to know that this involves the basic ideas of the classical calculus of variations–the Hamilton-Jacobi equation, extremal fields in the sense of Hilbert and Carathéodory, etc.

In the past, differential geometry has maintained a mutually profitable interchange of ideas with physics. Nowadays, physicists are more interested in indefinite Riemannian metrics. Again, the authors' narrow viewpoint has precluded anything on this topic. In fact, in parallel with this work–and with almost no interdisciplinary communication–physicists interested in applications of general relativity to cosmology and astrophysics have used similar techniques to prove singularity theorems about nonpositive Riemannian metrics. See the book, *The large-scale structure of space-time*, by Hawking and Ellis.

This book is a historical landmark in the sense that it is the first to concentrate on the successes of post-World War II differential geometry. Every book published before has been more-or-less an attempt to understand the work of the great masters in the light of the modern sensibility. Above all else, we have had to struggle to understand Elie Cartan! (One can even trace the spirit of this book back to Cartan, particularly *Géométrie des éspaces de Riemann*.) As I have already mentioned, the great successes recounted here have been achieved at the expense of partially, and perhaps only temporarily, abandoning the sweeping outlook of the classical work. One has only to compare this material to that in the collected works of Cartan and Lie and in Darboux' *Théorie des surfaces* to realize how much of our heritage has been dumped overboard. Perhaps this is due to our overemphasis on maintaining our status in the eyes of our big brothers, the topologists. I recall that when I was a student in the fifties everyone almost went around chanting, in Red Guard fashion: global good, local bad. Of course, this fanaticism had the happy consequence of leading to this impressive work; one will never know what might have been achieved if differential geometry had kept its traditional orientation. I would now want to ask how the techniques so precisely and powerfully developed in this modern work can be applied to the broader classical problems. My own guess is that most likely the field awaiting conquest is the geometric theory of nonlinear partial differential equations. (For example, it is not at all well known that Darboux' formidable treatise contains much more about this subject than it does about the theory of surfaces!) Perhaps the seeds to great advances in this field–and the recent discovery of "solitons" suggests that it is also of great interest for physics–lie in Darboux just as the seeds that grew to these magnificent comparison theorems were buried in the work of Jacobi, Riemann and Ricci.

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Discrete-parameter martingales, by Jacques Neveu, North-Holland, Amsterdam; American Elsevier, New York, 1975, viii + 236 pp., $26.95.

It is very simple to define a martingale. If $\mathfrak{T}_1 \subset \mathfrak{T}_2 \subset \ldots$ is an increasing sequence of sub σ-fields of the σ-field $\mathfrak{T}$ in a probability space $(\Omega, \mathfrak{T}, P)$, a sequence $\{x_n\}$ of real random variables is called *adapted* if x_n is $\mathfrak{T}_n$ measurable for $n = 1, 2, \ldots$. The adapted sequence is further called a supermartingale if $E(x_{n+1}|\mathfrak{T}_n) \leqslant x_n$, $n \geqslant 1$. It is called a martingale if the inequality is replaced by an equality and a submartingale if the inequality is reversed. Just one more definition is effectively all that is needed. A positive integer