variation. Itô's definition is obtained as a limit of step function approximations which works when (i) f is $\mathcal{B} \times \mathcal{F}$ measurable; (ii) for each $t \in [a, b]$, $f(t) \in L^2(\Omega, P)$ and $\int_a^b E(|f(t)|^2)\, dt < \infty$; (iii) for each $t \in [a, b]$, $f(t, \cdot)$ is measurable $\mathcal{F}(t)$, the σ-field generated by $\{B(s), s \leqslant t\}$. Note that condition (ii) restricts the average size of $|f(t)|$ and (iii) says that the dependence of $f(t, \omega)$ on ω is restricted to information about the past and present values of $B(s, \omega)$. This chapter does no more than give a taste of a large subject with important applications. An interested reader would go on to consult the book by McKean [**4**].

The reviewer enjoyed his commision to read the book. He suspects that the book will have limited value as a reference work because no topic is pushed very far. It does have a good selection of examples worked out in the text as well as problems at the end of each chapter, which are provided with outline solutions. This means that a competent graduate student or an analyst unfamiliar with stochastic processes would profit greatly by careful study of the book. It would make a good text for an advanced graduate course provided the lecturer was satisfied with the topics selected. The authors have provided a valuable new perspective on a variety of important analytic tools used for the study of stochastic processes.

References

1. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953. MR **15**, 445.
2. I. I. Gihman and A. V. Skorohod, *Introduction to the theory of random processes*, "Nauka", Moscow, 1965; English transl., Saunders, Philadelphia, Pa., 1969. MR **33** #6689; **40** #923.
3. P. Lévy, *Théorie de l'addition des variables aléatoires*, 2nd ed., Gauthier-Villars, Paris, 1954.
4. H. P. McKean, Jr., *Stochastic integrals*, Probability and Math. Statist., no. 5, Academic Press, New York, 1969. MR **40** #947.

S. J. Taylor

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

Homotopy theory; an introduction to algebraic topology, by Brayton Gray, Academic Press, New York, 1976, xiii + 368 pp., $22.00.

"This book is an exposition of elementary algebraic topology from the point of view of a homotopy theorist." It is with this sentence that the Preface to Brayton Gray's book begins, so perhaps we would be well advised to learn something of the homotopy theorist's point of view before examining the contents of the book itself.

In a vague sense homotopy theory studies properties of topological spaces that remain invariant under a continuous deformation. The achievements of the theory stem from the fact that so many seemingly rigid problems are really homotopy theoretic in nature.

Around the turn of the century, during the formative period of algebraic topology, Poincaré introduced [**9**] (among other things) the homology (groups) of a polyhedron. A polyhedron is a configuration of basic convex sets called simplexes, and it was from the combinatorial properties of the configuration that the homology of a polyhedron was defined. It then became essential to demonstrate that these combinatorially defined invariants were in fact independent of the particular combinatorial structure and depended only on the topological type of the polyhedron; that is homeomorphic polyhedra should have isomorphic homology. This problem gave rise to the Hauptvermutung (fundamental hypothesis): homeomorphic polyhedra admit isomorphic combinatorial structures. This is true for polyhedra of dimension at most three, and false in every dimension greater than four [**1**], [**2**]. It is true for simply connected combinatorial manifolds of dimension at least six, [**3**] (more than this is known but not germaine here) but certainly was not the means whereby the topological invariance of homology was established.

It was Alexander [**4**] who proved the topological invariance of homology, and this he did in a really suprising way. Let us say that two maps (a map is a continuous function) f, g: $X \to Y$ are *homotopic* iff there is a continuous family of maps φ_t: $X \to Y$, $0 \leqslant t \leqslant 1$, with $\varphi_0 = f$ and $\varphi_1 = g$, in which case we write $f \sim g$. It is easily verified that $\sim$ is an equivalence relation, so it is natural to say a map f: $X \to Y$ is a *homotopy equivalence* iff there exists a map $g: Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$. If there is a homotopy equivalence $X \to Y$ we say that X and Y have *the same homotopy type*; homeomorphic spaces are clearly of the same homotopy type. What Alexander proved was that polyhedra of the same homotopy type had isomorphic homology. Thereby not only establishing the topological invariance of homology, but more importantly both generalizing and clarifying (a not too common simultaneous occurrence!) the Hauptvermutung by establishing homotopy invariance as the key property of homology. From that day forward the basic functors of algebraic topology have been those that are homotopy type invariants.

Perhaps the oldest and most basic such functor is the fundamental group $\pi_1(X, x_0)$ of a topological space X with basepoint $x_0 \in X$. The higher homotopy groups $\pi_n(X, x_0)$, $n \geqslant 2$, were originally dismissed as being of no interest, since unlike the fundamental group they were always abelian. However Hopf's startling discovery [**6**] that $\pi_3(S^2, *) > \mathbf{Z}$ (in particular that there is a map h: $(S^3, *) \to (S^2, *)$ that is not homotopic to the constant map to the basepoint) and Freudenthal's theorem that $\pi_{n+k}(S^n, *)$ does not depend on n for $n > k$, revived interest in these groups. Despite the efforts of numerous workers, very little more was discovered about these groups until Serre [**10**] reunited the divergent developments of homotopy and homology and generated the extraordinary growth of algebraic topology during the past 25 years.

What distinguishes the homotopy theorist's viewpoint is the basic fact that all functors should be homotopy invariants *by their very definition*. Thus the fundamental theorem from which all else flows is the homotopy excision theorem. Once this is established, various important exact sequences and the Freudenthal suspension theorem quickly follow and homology/cohomology can be introduced as homotopy classes of maps into specially chosen spaces. By contrast, in a classical treatment of basic algebraic topology, exactness and computability for polyhedra of the homology functors are built into the definitions, and it is the homotopy invariance that is hard to establish. The Freudenthal suspension theorem, homotopy excision theorem etc., if treated at all, are deduced from the Serre exact sequence of a fibre space and the Whitehead theorem.

To motivate the study of homotopy invariant functors, Gray's book begins with some classical problems that are then reduced to homotopy theory. For example, if $\mathbf{B}^n \subset \mathbf{R}^n$ is the unit ball $\{x \in \mathbf{R}^n | \; \|x\| \leqslant 1\}$ it is required to determine if there exists a map f: $\mathbf{B}^n \hookleftarrow$ without fixed points, that is, with $f(x) \neq x \;\forall x \in \mathbf{B}^n$. Through a sequence of elementary deductions this is reduced to determining if $\pi_{n-1}(S^{n-1}, *) = 0$ or not, and the book proper begins naturally enough with the study of the fundamental group. Covering spaces are introduced as one means of computing these groups. The treatment of covering spaces is rather brisk, and the basic existence question is not considered at all. It is in this section (§7) that one meets Seminar Problems for the first time (though in later sections this nicety between seminar problems and exercises is not maintained; and some of the exercises are ridiculously hard). Here they consist of some quick definitions, and inappropriate reference to the literature, and a suggestion that the reader delve into the basic existence question. This is a rather unfortunate omission as the Galois correspondence between coverings of a (nice) space X and subgroups of the fundamental group $\pi_1(X; x_0)$ is a beautiful example of a homotopy theoretic solution to a classification problem. Though such completeness of classification is seldom, if ever, achieved it serves well as a model for many similar phenomena.

After discussing track groups (this is done in (9.2) and though track groups figures in the title of the chapter, nowhere can I actually find a *definition* of track groups) and the higher homotopy groups, we come to what might best be called the fundamental theorem of homotopy theory, namely the theorem of Blakers and Massey [2]: If $[X; X_1, X_2]$ is an excisive triad, and the pair $(X_1, X_1 \cap X_2)$ is $n - 1$ connected while $(X_2, X_1 \cap X_2)$ is $m - 1$ connected, then the natural map

$$i_*: \pi_r(X_1, X_1 \cap X_2; *) \to \pi_r(X, X_2; *)$$

is an isomorphism for $r < m + n - 2$, and surjective for $r = m + n - 2$. A first approximation to this theorem is given in §13 where the basic notion of attaching a cell is introduced, and the general formulation is to be found three sections later. These three sections and an appendix contain a great wealth of material beautifully expounded. Here one finds the basic constructions of homotopy theory; mapping cones, mapping cylinders etc., the homotopy theory of *cw* complexes, culminating in the Blakers-Massey theorem, which are then applied to solve the classical problems (is there a fixed point free self map of the n-ball, etc.), derive the Freudenthal and Hopf theorems and numerous other important results. This brings to a close what might be considered as the first, or classical homotopy theory, portion of the book.

Lest one come away with the impression (after all 16 sections have gone by) that nothing has happened for hundreds of pages let it be noted that §16 ends on page 156, and in addition to the fundamental theorems of homotopy theory, fibre bundles and simplicial complexes have been introduced, and some of their elementary properties established.

Now begins the second theme of the book, devoted to the introduction of homology and cohomology theories. Historically, as already remarked, homology and cohomology were first defined for polyhedra by an explicit algebraic construction depending on the combinatorial decomposition of the polyhedron. The question of the topological invariance of these objects was cloudy and difficult. Gray's approach is by and large dual (speaking of which, the treatment of Poincaré duality is of such generality as to incompass Spanier-Whitehead duality and perhaps even Ekmann-Hilton duality). A functor from spaces to groups is introduced which by its very definition is homotopy invariant, and on the family polyhedra it is verified that the functor is computable by a combinatorial formula. Thus Moore spaces, Eilenberg-Mac Lane spaces and Postnikov towers are introduced, followed by spectra and homology and cohomology theories with coefficients in a spectrum. This awesome generality may appal some classicists, but it is well done, and I doubt anymore confusing than the mass of indices, subscripts and multiple summation formulae one subjects students to in a classical treatment of simplicial homology. The additive properties of homology are brought to a close with a proof of the Hurewicz theorem. This is followed by a long section (§24) on multiplicative structures, in which one will (at least I *think* one will) find every conceivable pairing, associativity and commutativity formula one could want. One will also find (p. 231) what is probably one of the largest commutative diagrams ever to reach print (containing several misprints, the W in the 2nd column should be Y and the 2nd E in the last entry an F) from which the various multiplicative structures are defined. I must confess I found this less than edifying. Another rather extraordinary commuting diagram appears in the next section (p. 249) where much of the multiplicativity done for spectra is there translated to the language of chain complexes, and which culminates with the classical Künneth and Universal coefficient theorems.

The remaining sections treat Poincaré duality (§26), cohomology operations (§§27, 28), K theory and cobordism theory. The treatment of cohomology operations is particularly nice, and there is a hard to decipher table of Adem relations for Sq^aSq^b, $a < 2b \leqslant 12$, at the end of the book. The chapters on K theory and cobordism theory are very abbreviated and sketchy, and the space they occupy could have been better used to fill in some of the details relegated to the exercises in earlier chapters. They also contain a serious (though common) misstatement of the structure of the complex cobordism ring: the complex projective spaces are fine polynomial generators over the rationals, but they simply will not do over the integers. It is regrettable that these pages were not used earlier in the book to motivate and smooth the introduction of spectra. The Freudenthal suspension theorem shows that the *conditional* exactness of the homotopy Mayer-Vietoris sequence can be replaced by a long exact sequence if one passes to the suspension category. The transition to naive spectra is then almost painless.

Throughout the book the exposition is good, with many nice and interesting proofs for what are often musty old theorems. The level of generality employed is appropriate to the material presented, is not overdone and bogged down in an overly complex notation (except perhaps the chapters on products where it seems unavoidable), and there is an excellent index of symbols which is a great aid in using the book as a reference. The early part of the book is particularly well motivated, but off towards the middle of the book, new concepts are introduced without any clear explanation of why. This is a typical failing of text books: to suppose that after a while the material motivates itself. A more serious objection, particularly for one contemplating self study or using the book by itself in a course, are the rather large number of results relegated to the exercises in section k, only to resurface as essential to the proof of a theorem in section $n + k$, where $n + k$ is often in the stable range, i.e., $n > k$.

Nevertheless, Gray's book provides an elegant view of basic algebraic topology contrasting very nicely with the classical viewpoint of Eilenberg-Steenrod [4], and I for one feel that the two together provide a superb basis for either a course or self study.

For example the first 16 sections of G could be profitably followed by the Introduction and Chapter I of E-S. Motivation for the axioms can be found by passing to the stable range on the homotopy groups functors to make conditional theorems (homotopy excision, Freudenthal, etc.) unconditional; or by introducing manifolds and de Rham theory as for example in [11]. Then back to G for §§17–20 followed by E-S II–IV, XI whence back to G for §§21–25, at which point a very solid foundation for further study in algebraic topology will have been laid down.

In recent years many introductory algebraic topology textbooks have become available. In comparing Gray's book to these it comes off very well indeed. It does not get overly involved in definitions of polyhedra, and/or complexes, fibrings/cofibrings, or local topological properties. It certainly is not overlong, and does have a good index of notation. It is written well enough that most reasonably advanced students should be able to open it anywhere, and with a minimum of cross references start having it make good sense inside of 15 minutes. In short, it is an excellent book.

References

1. J. W. Alexander, *A proof of the invariance of certain constants of analysis situs*, Trans. Amer. Math. Soc. **16** (1915), 148–154.

2. A. L. Blakers and W. S. Massey, *The homotopy groups of a triad.* II, Ann. of Math. (2) **55** (1952), 192–201. MR **13**, 485.

3. H. Cartan, *Séminaire de l'Ecole Normale Supérieure*, 1954/55, *Algèbres d'Eilenberg-Mac Lane et homotopie*, Secrétariat mathématique, Paris, 1955. MR **19**, 438.

4. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N. J., 1952. MR **14**, 398.

5. H. Freudenthal, *Über die Klassen von Spharenabbildungen*, Compositio Math. **5** (1937), 299–314.

6. H. Hopf, *Über die Abbildungen de driedimensionalen Sphere auf de Kugelfläche*, Math. Ann. **104** (1931), 637–665.

7. W. Hurewicz, *Beiträge zur Topologie de Deformationen.* I, *Hoherdimensionale Homotopiegruppen*, Prok. Akad. Wetensch. Amsterdam **38** (1935), 112–119.

8. J. W. Milnor, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math. (2) **74** (1961), 575–590. MR **24** #A2961; erratum, **25**, p. 1242.

9. H. Poincaré, *Analysis situs*, J. Ecole Polytech. Paris (2) **1** (1895), 1–123.

10. J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505. MR **13**, 574.

11. I. M. Singer and J. A. Thorpe, *Lecture notes on elementary topology and geometry*, Scott, Foresman, Glenview, Ill., 1967. MR **35** #4834.

12. J. R. Stallings, *On infinite processes leading to differentiability in the complement of a point*, Differential and Combinatorial Topology (A Sympos. in Honor of Marston Morse), Princeton Univ. Press, Princeton, N. J., 1965, pp. 245–254. MR **31** #5213.

13. D. Sullivan, *Geometric topology*, Seminar Notes, Princeton Mimeo, 1967.

LARRY SMITH