BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

# THE AMALGAMATED FREE PRODUCT STRUCTURE OF $\mathrm{GL}_2(K[X_1, \ldots, X_n])$

BY DAVID WRIGHT

Communicated by Hyman Bass, May 11, 1976

For any ring $R$ we let $E_2(R)$ be the subgroup of $\mathrm{GL}_2(R)$ generated by elementary matrices. We let $E_2'(R)$ be the subgroup of $\mathrm{GL}_2(R)$ generated by $E_2(R)$ and the invertible diagonal matrices. We denote by $B_2(R)$ the lower triangular subgroup of $\mathrm{GL}_2(R)$.

The following classical theorem is due to Nagao [2].

THEOREM 1. *Let $K$ be a field, $X$ an indeterminate. Then*

$$\mathrm{GL}_2(K[X]) = \mathrm{GL}_2(K) *_{B_2(K)} B_2(K[X]).$$

More generally, we have

THEOREM 2. *Let $R$ be an integral domain, and $X_1, \ldots, X_n$ indeterminates. Then*

$$E_2'(R[X_1, \ldots, X_n]) = E_2'(R) *_{B_2(R)} B_2(R[X_1, \ldots, X_n]).$$

(This generalizes Theorem 1, since $E_2'(R) = \mathrm{GL}_2(R)$ if $R = K$ or $R = K[X]$, $K$ a field.)

Now let $K$ be a field, and $X_1, \ldots, X_n$ indeterminates. The group $\mathrm{GL}_2(K[X_1, \ldots, X_n])$, $n > 1$, is more difficult to understand because it is not generated by diagonal and elementary matrices (see [1]). However, the following technical lemmas enable us to describe $\mathrm{GL}_2(K[X_1, \ldots, X_n])$ as a certain free product with amalgamation (see Theorem 3 below).

Let $G$ be a group with subgroups $A$ and $C$ such that $G = A *_B C$, where $B = A \cap C$. Let $I$ (resp. $J$) be systems of nontrivial left coset representatives of $A$ (resp. $C$) modulo $B$. With respect to these choices, any element of $G$ has a unique normal form (see [2, Chapter I]). Given a subgroup $H \subset G$, we let $A_H = A \cap H$, $B_H = B \cap H$.

LEMMA 1. *Suppose there is a retraction $r$: $G \longrightarrow A$, and suppose $H$ is a subgroup of $G$ such that $r(H) = A_H$, and such that $A_H$ acts transitively on $A/B$. Then, letting $C' = C \cap r^{-1}(A_H)$, we have $r^{-1}(A_H) = A_H *_{B_H} C'$ ($\supset H$).*

---

*AMS (MOS) subject classifications* (1970). Primary 20H05; Secondary 15A21, 20F55, 13C99.

LEMMA 2. *Suppose $H$ is a subgroup of $G$ containing $A$. Let $\mathcal{W}$ be the collection of all elements $h \in H$ of the form $h = c_1 a_1 \cdots c_{t-1} a_{t-1} c_t$, $t \geqslant 0$, with $a_1, \ldots, a_{t-1} \in I$, $c_1, \ldots, c_t \in J$, such that $c_1 a_1 \cdots c_{s-1} a_{s-1} c_s \notin H$ if $s < t$. Then $W = B\mathcal{W}$ is a subgroup of $H$. Furthermore, $B = A \cap W$ and $H = A *_B W$. Clearly, $W \supset H \cap C$. The subgroup $W$ is independent of the choices of $I$ and $J$.*

THEOREM 3. *Let $R = K[X_1, \ldots, X_n]$ with $K$ a Euclidean domain. Then $\mathrm{GL}_2(R)$ is the free product of $E_2'(R)$ with a subgroup $W = W(K)_{(X_1,\ldots,X_n)}$, amalgamated along the intersection $E_2'(R) \cap W = B_2(R)$. The inclusion $B_2(R) \subset W$ is strict unless $R$ is a Euclidean domain.*

(As the notation $W = W(K)_{(X_1,\ldots,X_n)}$ suggests, and as the proof will indicate, $W$ canonically depends on the choice and ordering of the variables $X_1, \ldots, X_n$.)

A complete version of these statements and their proofs will appear later. For now, we sketch the proof of Theorem 3, using Lemmas 1 and 2. For $n = 1$ and $K$ a field, we satisfy the theorem trivially by taking $W = B_2(K[X])$. We will now show that if the theorem holds for a fixed integer $n \geqslant 1$ when $K$ is a field, then it is true when $K$ is any Euclidean domain. In particular, if $K = F[X_1]$, $F$ a field, then we set $W(F)_{(X_1,\ldots,X_{n+1})} = W(K)_{(X_2,\ldots,X_{n+1})}$, and so the theorem will be proved inductively.

Let $K$ be a Euclidean domain, $F$ its field of fractions, and $R = K[X_1, \ldots, X_n]$; and assume the theorem holds for $F[X_1, \ldots, X_n]$. Upon letting $G = \mathrm{GL}_2(F[X_1, \ldots, X_n])$; $A = \mathrm{GL}_2(F)$; $C = W(F)_{(X_1,\ldots,X_n)}$; and $B = B_2(F)$, we apply our assumption and Theorem 2 to get $G = A *_B C$. We now appeal to Lemma 1, letting $H = \mathrm{GL}_2(R)$, and $r\colon G \longrightarrow A$ be induced by setting $X_1 = \cdots = X_n = 0$. Now, $\mathrm{GL}_2(K)$ acts transitively on $\mathrm{GL}_2(F)/B_2(F)$, and so we get

$$\mathrm{GL}_2(R) \subset \mathrm{GL}_2(K) *_{B_2(K)} C' = r^{-1}(\mathrm{GL}_2(K))$$

where $C' = C \cap r^{-1}(\mathrm{GL}_2(K))$.

Clearly $B_2(R) \subset C' \cap H$. We now apply Lemma 2 with $A = \mathrm{GL}_2(K)$; $C = C'$; $B = B_2(K)$; $G = r^{-1}(\mathrm{GL}_2(K))$ (hence $G = A *_B C$); and $H = \mathrm{GL}_2(R)$ to get the subgroup $W = W(K)_{(X_1,\ldots,X_n)}$ containing $B_2(R)$ such that

$$\mathrm{GL}_2(R) = \mathrm{GL}_2(K) *_{B_2(K)} W = \mathrm{GL}_2(K) *_{B_2(K)} B_2(R) *_{B_2(R)} W$$

$$= E_2'(R) *_{B_2(R)} W.$$

(The last step appeals to Theorem 2.)

REMARK 1. For $H \subset \mathrm{GL}_2(R)$ we let $SH = H \cap \mathrm{SL}_2(R)$. A slight modification of the above proof shows that, for $R$ and $W$ as in Theorem 3, $\mathrm{SL}_2(R) = E_2(R) *_{SB_2(R)} SW$.

REMARK 2. For any ring $R$ we define $GA_2(R)$ to be $\mathrm{Aut}_R(R[X, Y])$.

The methods used to prove Theorem 3 also show that $GA_2(K[X_1, \ldots, X_n])$ has a somewhat similar free product decomposition, for $K$ a Euclidean domain.

REFERENCES

1. P. M. Cohn, *On the structure of the* $\mathrm{GL}_2$ *of a ring,* Hautes Études Sci. Publ. Math. **30** (1966), 5–53. MR **34** #7670.

2. H. Nagao, *On* GL(2, $K[x]$), J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 117–121. MR **22** #5684.

3. J.-P. Serre, *Arbres, amalgames, et* $Sl_2$, Collège de France 1968/69, Lecture Notes in Math., Springer-Verlag, Berlin and New York (to appear).

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63103