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ELLIPTIC PSEUDO DIFFERENTIAL OPERATORS DEGENERATE ON A SYMPLECTIC SUBMANIFOLD

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1. **Introduction.** This note is concerned with the classes of pseudo differential operators $L^{m,M}(\Omega, \Sigma)$, Σ symplectic submanifold of codimension 2, in Sjöstrand **[4]**; the definitions of P in $L^{m,M}(\Omega, \Sigma)$ and of the associated winding number N are recalled in §2. In Helffer **[2]** the study of the hypoellipticity of P is reduced to the analysis of the bounded solutions of an ordinary differential equation. Here we deduce an explicit result for $N = 2 - M$: essentially, we can prove that in this case all the bounded solutions are products of an exponential function with polynomials.

2. **The classes $L^{m,M}(\Omega, \Sigma)$ and the winding number.** Let $\Omega \subset \mathbf{R}^n$ be an open set. Let $\Sigma \subset T^*(\Omega)\backslash 0$ be a closed conic symplectic submanifold of codimension 2 (Σ symplectic means that the restriction of the symplectic form $\omega = \Sigma d\xi_s \wedge dx_s$ to Σ is nondegenerate). $L^{m,M}(\Omega, \Sigma)$ is the set of all the pseudo differential operators P which have a symbol of the form

$$(1) \qquad p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j/2}(x, \xi),$$

where $p_{m-j/2}$ is positively homogeneous of degree $m - j/2$ and for every $K \subset\subset \Omega$ there exists a constant C_K such that

$$(2) \qquad |p_m(x, \xi)|/|\xi|^m \geqslant C_K^{-1} d_\Sigma^M(x, \xi),$$

$$(3) \qquad |p_{m-j/2}(x, \xi)|/|\xi|^{m-j/2} \leqslant C_K d_\Sigma^{M-j}(x, \xi), \qquad 0 \leqslant j \leqslant M,$$

for all $(x, \xi) \in K \times \mathbf{R}^n$, $|\xi| > 1$ ($d_\Sigma(x, \xi)$ is the distance from $(x, \xi/|\xi|)$ to Σ).

Fix ρ in Σ, denote by $N_\rho(\Sigma)$ the orthogonal space of $T_\rho(\Sigma)$ with respect to ω and choose two linear coordinates on $N_\rho(\Sigma)$ u_1, u_2 such that $\omega/N_\rho(\Sigma) = du_2 \wedge du_1$. Take $X = (u_1, u_2) \in N_\rho(\Sigma)$ and let V be any vector field on $T^*(\Omega)$ equal to X at ρ. We define the homogeneous polynomial

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$$c \prod_{h=1}^{M} (u_2 - r_h u_1) = \frac{1}{M!} (V^M p_m)_\rho. \tag{4}$$

In view of (2) $\operatorname{Im} r_h \neq 0$ for each h, $h = 1, \ldots, M$: let M^+ (M^-) be the number of the r_h's such that $\operatorname{Im} r_h > 0$ ($\operatorname{Im} r_h < 0$). The integer $N = M^+ - M^-$ may take the values $M, M-2, \ldots, 2-M, -M$; here we assume $M \geqslant 2$ and

$$N = 2 - M, \textit{ for every } \rho \textit{ in } \Sigma. \tag{5}$$

3. **The problem of the hypoellipticity.** Let $P \in L^{m,M}(\Omega, \Sigma)$ satisfy (5). We are interested in the following hypoellipticity property:

(6) *For any open subset U of Ω and any distribution f in U, $Pf \in H^s_{\mathrm{loc}}(U)$ implies $f \in H^{s+m-M/2}_{\mathrm{loc}}(U)$.*

Let $\mathcal{P}$ be the algebraic vector space of all the polynomials in one real variable with complex coefficients and denote by $L(\mathcal{P})$ the space of all the linear maps from $\mathcal{P}$ into $\mathcal{P}$. We associate to P an application A_P: $\rho \in \Sigma \longrightarrow A_P(\rho) \in L(\mathcal{P})$. The explicit definition of A_P will be given in §4; first let us state our main result.

THEOREM 1. *Let $P \in L^{m,M}(\Omega, \Sigma)$ satisfy* (5). *Then* (6) *holds if and only if*

$$\text{dimension Ker } A_P(\rho) = 0, \textit{ for every } \rho \textit{ in } \Sigma. \tag{7}$$

4. **Definition of $A_P(\rho)$.** If (5) is satisfied, then it is $M^+ = 1$ and $M^- = M - 1$: we will assume $\operatorname{Im} r_h < 0$ for $2 \leqslant h \leqslant M$ and $\operatorname{Im} r_1 > 0$. As in Helffer [2], initially we construct a family of ordinary differential operators with polynomial coefficients. Consider the symbol $q(x, \xi)$ with asymptotic expansion

$$\sum_{j=0}^{\infty} q_{m-j/2} \sim \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \left(\sum \frac{1}{2i} \frac{\partial^2}{\partial x_s \partial \xi_s} \right)^t p.$$

Using the notations of §2, we define on $N_\rho(\Sigma)$ the polynomial (the leading part coincides with (4))

$$c \prod_{h=1}^{M} (u_2 - r_h u_1) + \sum_{\alpha+\beta<M} c_{\alpha,\beta} u_1^\alpha u_2^\beta = \sum_{j=0}^{M} \frac{1}{(M-j)!} (V^{M-j} q_{m-j/2})_\rho. \tag{8}$$

We rewrite the left-hand side of (8) in the symmetric form

$$\sum_{\gamma(h), h \leqslant M} c'_{\gamma(h)} u^{\gamma(h)}, \tag{9}$$

where the components of the multiorder $\gamma(h) = (\gamma_1, \ldots, \gamma_h)$ may take the value 1 or 2, $c'_{\gamma(h)} = c'_{\delta(h)}$ if $|\gamma(h)| = |\delta(h)|$ and we have noted

$$u^{\gamma(h)} = u_{\gamma_1} u_{\gamma_2} \cdots u_{\gamma_h}. \tag{10}$$

Now, maintaining the order of the factors in (10), we replace u_2 in (9) by $D = -id/du_1$. We get a differential operator $\mathcal{M}(\rho)$ which can be expressed in the form

$$\mathcal{M}(\rho) = c(D - r_M u_1) \cdots (D - r_1 u_1) + \sum_{\alpha+\beta<M} c''_{\alpha,\beta} u_1^\alpha D^\beta. \tag{11}$$

Set

$$\eta = -i \sum_{\alpha+\beta=M-1} c''_{\alpha,\beta} r_1^\beta \Big/ c \prod_{h=2}^{M} (r_1 - r_h). \tag{12}$$

We define for $Q \in \mathcal{P}$,

$$\begin{aligned} A_{\mathcal{P}}(\rho)Q(u_1) &= \exp(-ir_1u_1^2/2 - \eta u_1) \\ &\quad \cdot \mathcal{M}(\rho)\,[\exp(ir_1u_1^2/2 + \eta u_1)Q(u_1)]. \end{aligned} \tag{13}$$

The definition of $A_{\mathcal{P}}(\rho)$ depends on the initial choice of the coordinates u_1, u_2. We can prove that, starting from other canonical coordinates u'_1, u'_2 and repeating the construction, we get a map $A'_{\mathcal{P}}(\rho)$ such that $U^{-1}(\rho)A'_{\mathcal{P}}(\rho)U(\rho) = A_{\mathcal{P}}(\rho)$ for some automorphism $U(\rho)$ in $L(\mathcal{P})$. Therefore condition (7) has an invariant meaning.

5. **Applications.** Take $Q(u_1) = \Sigma_{\nu=0}^k b_\nu u_1^\nu$. Developing (13) we obtain

$$A_{\mathcal{P}}(\rho)Q(u_1) = \sum_{\mu=0}^{k+M-2} \left(\sum_{\nu=0}^{k} d_{\mu,\nu} b_\nu \right) u_1^\mu, \tag{14}$$

where $d_{\mu,\nu}$ are polynomials in the variables $c, r_1, \ldots, r_M, c''_{\alpha,\beta}$. We write $\mathcal{D}^{(k)}$ for the matrix $(d_{\mu,\nu})$, $\mu = 0, \ldots, k + M - 2$, $\nu = 0, \ldots, k$. Let $\sigma = \{\mu_0, \mu_1, \ldots, \mu_k\}$ be a subset of $\{0, 1, \ldots, k + M - 2\}$ and let $\mathcal{D}_\sigma^{(k)}$ denote the minor $(d_{\mu_t,\nu})$, $t = 0, \ldots, k$, $\nu = 0, \ldots, k$. Theorem 1 can be rewritten in the following way.

THEOREM 2. *Let $P \in L^{m,M}(\Omega, \Sigma)$ satisfy* (5). *Then* (6) *holds if and only if for each fixed $\rho \in \Sigma$ and for every integer $k \geq 0$ there exists a subset $\sigma = \{\mu_0, \mu_1, \ldots, \mu_k\}$ of $\{0, 1, \ldots, k + M - 2\}$, such that*

$$\det \mathcal{D}_\sigma^{(k)} \neq 0. \tag{15}$$

A direct computation shows that for $\sigma_0 = \{M - 2, M - 1, \ldots, k + M - 2\}$

$$\det \mathcal{D}_{\sigma_0}^{(k)} = \lambda(\rho) \prod_{\nu=0}^{k} [\mathcal{L}(\rho) - \nu],$$

where $\lambda(\rho)$, $\mathcal{L}(\rho)$ are rational functions of $c, r_1, \ldots, r_M, c''_{\alpha,\beta}$: $\lambda(\rho) \neq 0$ and $\mathcal{L}(\rho)$ coincides with the invariant in Boutet de Monvel and Treves [1] and Helffer [3].

REFERENCES

1. L. Boutet de Monvel and F. Trèves, *On a class of pseudodifferential operators with double characteristics,* Invent. Math. 24 (1974), 1–34. MR 50 #5550.

2. B. Helffer, *Invariants associés à une classe d'opérateurs pseudodifférentiels et applications à l'hypoellipticité*, Ann. Inst. Fourier (Grenoble) (to appear).

3. ———, *Sur une classe d'opérateurs hypoelliptiques à caractéristiques multiples,* J. Math. Pures Appl. (to appear).

4. J. Sjöstrand, *Parametrices for pseudodifferential operators with multiple characteristics*, Ark. Mat. 12 (1974), 85–130, MR 50 #5236.

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