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INFINITE LOOP MAPS AND THE COMPLEX J-HOMOMORPHISM

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ABSTRACT. We study the complex J-homomorphism $j: U \to SG$ as the composition of two infinite loop maps.

1. **Introduction.** Let p be an odd prime and let q be a prime generating the units of Z/p^2. All spaces will be p-localized. The solution of the Adams conjecture establishes a commutative diagram of fibre sequences.

$$\begin{array}{ccccccccc} \cdots \longrightarrow U & \xrightarrow{\psi^q - 1} & U & \xrightarrow{\omega} & J^{\oplus} & \longrightarrow & BU^{\oplus} & \xrightarrow{\psi^q - 1} & BU^{\oplus} \\ & & \| & & \downarrow \mu & & \downarrow \tau & & \| \\ \cdots\cdots\cdots & \longrightarrow & U & \xrightarrow{j} & SG & \longrightarrow & SG/U & \longrightarrow & BU^{\oplus} \end{array} \tag{1.1}$$

Several, possibly different, τ have been constructed ([2], [5] and [8]). Given τ, then μ is unique. The fibre sequences are sequences of infinite loop maps and it is natural to ask whether (1.1) can be extended arbitrarily to the right—the infinite loop Adams conjecture. By [4] this would be true if τ were an infinite loop map. These results suggest strongly the validity of the conjecture.

In [2] an H-map, τ, is given. If $\mathbf{F}_q$ is the field with q elements the finite dimensional vector spaces over $\mathbf{F}_q$ under direct sum form a permutative category from which the infinite loopspace $J^{\oplus}$ is constructed by the technique of [1]. Similarly SG is obtained from a category of finite sets under cartesian product. The forgetful functor gives the "discrete models" infinite loop maps $\delta: J^{\oplus} \longrightarrow SG$.

THEOREM 1. *If τ is the map constructed in* [2] *then $\mu = \delta$ in* (1.1).

$J^{\otimes}$ is the infinite loopspace obtained from a category of vector spaces of $\mathbf{F}_q$ under tensor product. Assigning to a set the vector space generated by its elements gives $\nu: SG \longrightarrow J^{\otimes}$. Define Coker $J^{\otimes}$ by the infinite loop fibering Coker $J^{\otimes} \xrightarrow{\pi} SG \xrightarrow{\nu} J^{\otimes}$.

THEOREM 2. *$\nu \circ f: J^{\oplus} \longrightarrow J^{\otimes}$ is a homotopy equivalence for any map $f: J^{\oplus} \longrightarrow SG$ such that $f_{\#}$ is nontrivial on π_{2p-3}.*

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THEOREM 3. *In* (1.1), $j = \delta \circ \omega$.

Combining this with Theorem 2 we easily obtain

THEOREM 4 [9]. *There is an equivalence of infinite loopspaces* $\delta + \pi$: $J^{\oplus} \times \text{Coker}\, J^{\otimes} \longrightarrow SG$.

2. If the infinite loop Adams conjecture were true then there would exist an infinite loop map $J^{\oplus} \longrightarrow SG$ satisfying Theorems 1, 3 and 4.

Theorems 2, 3 and 4 can be proved without mentioning τ at all, i.e. without the solution of the Adams conjecture. For example cf. [7, I].

PROOF OF THEOREM 3. In [6] a cohomology theory, Ad_q^* is constructed satisfying

$$[X, Z \times J^{\oplus}] = Ad_q^0(X)$$

and giving an infinite loopspace structure to $Z \times J^{\oplus}$ extending the usual one on $J^{\oplus}$. $Ad_q^0(X)$ has a description in terms of isomorphisms of Z/q-vector bundles

$$\theta: E^{\otimes q} \longrightarrow E \oplus (E' \otimes N)$$

where E, E' are complex vector bundles over X and N is the complex regular representation of Z/q. A similar theory constructed from isomorphisms, θ, such that

$$\mu(\theta): E \xrightarrow{\text{diag}} E^{\otimes q} \longrightarrow E \oplus (E' \otimes N) \xrightarrow{\text{proj}} E$$

is a proper map is also Ad_q^*. Sending θ to the stabilization of $\mu(\theta)$ gives an exponential H-map,

$$\mu: \bigcup_{n \geqslant 0} (n) \times J^{\oplus} \longrightarrow \bigcup_{n \geqslant 0} Q_{q^n} S^0,$$

where $Q_{q^n}S^0$ is the set of maps of degree q^n in $\Omega^\infty S^\infty$. It is easy to show explicitly that

$$\mu \circ \omega = j: U \longrightarrow (0) \times J^{\oplus} \longrightarrow Q_1 S^0 \longrightarrow SG.$$

Also Ad_q^0 has a transfer for cyclic coverings which admits an explicit bundle-theoretic description from which it is simple to see that μ commutes with cyclic covering transfers [3] of the two infinite loopspaces $J^{\oplus}$ and SG. Since δ extends to an exponential H-map

$$\delta: \bigcup_{n \geqslant 0} (n) \times J^{\oplus} \longrightarrow \bigcup_{n \geqslant 0} Q_{q^n} S^0$$

which commutes with transfers for finite coverings, Theorem 3 is a consequence of the following result.

THEOREM 5. *There is a unique exponential H-map* μ: $\bigcup_{n \geqslant 0}(n) \times J^{\oplus} \longrightarrow \bigcup_{n \geqslant 0} Q_{q^n} S^0$ *which commutes with p-fold cyclic covering transfers and maps* $(n) \times J^{\oplus}$ *to* $Q_{q^n}S^0$.

Since τ induces a unique μ we may also deduce Theorem 3 from Theorem 1, once we acknowledge the existence of τ.

PROOF OF THEOREM 1 (cf. [7, II]). τ is described explicitly in terms of the geometry of fibre bundles of the form $U(n)/N \longrightarrow BN \longrightarrow BU(n)$. The transfer on SG/U may be extended to the space $\bigcup_{n\geqslant 0}(n) \times (SG/U)$. Furthermore τ may be extended to an H-map

$$\bar{\tau}\colon \bigcup_{n\geqslant 0} (n) \times BU^{\oplus} \longrightarrow \bigcup_{n\geqslant 0} (n) \times (SG/U),$$

which maps $(n) \times BU^{\oplus}$ to $(n) \times (SG/U)$. Also $\bar{\tau}$ commutes with p-fold cyclic covering transfers. The proof of Theorem 1 is completed by means of the analogue of Theorem 5 for H-maps $\bigcup_{n\geqslant 0}(n) \times J^{\oplus} \longrightarrow \bigcup_{n\geqslant 0}(n) \times (SG/U)$.

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