STABLE ARCS OF DIFFEOMORPHISMS

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We consider here arcs of diffeomorphisms which, being initially structurally stable, go through "brusque" changes (or bifurcations) in their phase portrait (space of orbits) as the parameter changes. We are mainly interested in stable arcs in the sense that nearby arcs have the "same" changes in their phase portraits. As we shall see, there are various natural ways of defining stability for arcs. For a large class of arcs starting as Morse-Smale diffeomorphisms we characterize the arcs that are stable according to the various definitions of stability. We conjecture that this "large class" really contains all stable arcs of diffeomorphisms starting as Morse-Smale diffeomorphisms. We present below the precise statements of the results, beginning with some definitions and known facts. The proofs will appear elsewhere.

M denotes a compact C^∞ manifold without boundary. Diff(M) is the space of C^∞ diffeomorphisms of M and $\Phi(M)$ is the space of C^∞ arcs of diffeomorphisms on M. That is, $\Phi(M)$ consists of C^∞ mappings $f: M \times I \to M$, $I = [0, 1]$, such that for each $t \in I$, f_t defined by $f_t(m) = f(m, t)$ is a diffeomorphism. We give Diff(M) and $\Phi(M)$ the usual C^∞-topologies.

For $g \in$ Diff(M), the orbit $O(x)$ of a point $x \in M$ is defined by $O(x) = \{g^n(x) | n \in \mathbf{Z}\}$; a point $y \in M$ is called a limit point of g if for some $x \in M$ and sequence n_i, $i = 1, 2, \ldots$, $n_i \in \mathbf{Z}$, $|n_i| \to \infty$, $\lim g^{n_i}(x) = y$. The closure of the limit points of g is called the limit set of g and denoted by $L(g)$. A point $x \in M$ is a periodic point of g with period n if $g^n(x) = x$ and $g^r(x) \neq x$ for all $0 < r < n$; x is hyperbolic if $T_x(g^n)$ has no eigenvalues on the unit circle. The stable, unstable, sets or manifolds $W^s(x, g)$, $W^u(x, g)$, of a periodic point x are defined as $\{y \in M | \mathrm{dist}(f^n(y), f^n(x)) \to 0 \text{ for } n \to +\infty\}$ and $\{y \in M | \mathrm{dist}(f^n(y), f^n(x)) \to 0 \text{ for } n \to -\infty\}$ respectively. If x is a hyperbolic periodic point of g, $W^s(x, g)$ and $W^u(x, g)$ are smoothly immersed submanifolds of M. We say that a diffeomorphism $g \in$ Diff(M), with finite limit set, has an n-cycle if there is a sequence of periodic orbits $O(p_1), \ldots, O(p_n)$ with $O(p_0) = O(p_n)$ and $O(p_{i+1}) \subset \text{closure}(W^u(O(p_i)) - O(p_i))$ for $0 \leq i < n$. The cycle has length $n \geq 1$ if no other periodic orbit can be added to the previous sequence.

A diffeomorphism g is Morse-Smale if $L(g)$ consists of finitely many hyperbolic periodic points and if all the intersections of stable and unstable manifolds are transverse. We denote the set of Morse-Smale diffeomorphisms by M.S. This set is open [5] and each $g \in$ M.S. is stable in the sense that any $g_1 \in C^1$ close to g is topologically conjugate to g [5], [6], i.e., there is a homeomorphism h: $M \longrightarrow M$ such that $g_1 h = hg$.

For an arc $f \in \Phi$ with $f_0 \in$ M.S., let $b = b(f) = \inf\{t \in I | f_t \notin$ M.S.$\}$. We always assume that $b(f) < 1$, since any arc f_t with $f_t \in$ M.S. for all $t \in [0, 1]$ is stable. An arc $f \in \Phi$ with $f_0 \in$ M.S. is called stable if there are $\epsilon > 0$ and a neighborhood n of f in Φ such that, for any $\tilde{f} \in n$ there is a continuous injective map η: $[0, b(f) + \epsilon] \longrightarrow [0, 1]$ with $\eta(0) = 0$ and a map γ: $[0, b(f) + \epsilon]$ $\longrightarrow$ Homeo(M) such that $f_t \circ \gamma(t) = \gamma(t) \circ \tilde{f}_{\eta(t)}$ for all $t \in [0, b(f) + \epsilon]$. The arc is called continuously stable if it is stable and if it is possible to choose γ continuous in the above definition. It is called left stable if the maps η and γ as above exist and are continuous but only defined on $[0, b(f)]$.

Now we come to the description of the class of arcs in Φ to which our results apply; this class will be denoted by $\mathfrak{A}$.

DEFINITION. $\mathfrak{A} \subset \Phi$ is the subset of those arcs $f \in \Phi$ for which

(i) $f_0 \in$ M.S.;

(ii) $b = b(f) = \inf\{t \in I | f_t \notin$ M.S.$\} < 1$;

and

(a) the limit set of f_b is finite;

(b) if f_b has a saddle-node (see definition below) which is part of a cycle then this cycle has length bigger than one.

Many arcs satisfy these conditions. However, it would be desirable to be able to remove requirements (a) and (b). This would be the case if the following conjectures are true. In [3], [4] a description of the orbit structure of generic arcs in $\mathfrak{A}$ was given, and it was conjectured that for a generic arc $f \in \Phi$ with $f_0 \in$ M.S. and $b(f) < 1$, the limit set of $f_{b(f)}$ would have finitely many orbits. This conjecture with some work would imply that we could skip condition (a); but in our context even a weaker conjecture would be enough:

CONJECTURE 1. For any arc $f \in \Phi$, $f_0 \in$ M.S., which is left stable, $L(f_b)$ is finite.

Also, if the following conjecture of Arnold were true, we could skip condition (b).[2]

CONJECTURE 2 (ARNOLD [1]). There is a set $M \subset [0, 1]$ of measure 1 such that if φ: $S^1 \longrightarrow S^1$ is an analytic diffeomorphism with rotation number $\mu(\varphi) \in M$, then φ is analytically conjugate to a rotation over an angle $2\pi \cdot \mu(\varphi)$.

We need some further definitions related to the phenomena which occur at the first bifurcation point $b(f)$.

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[2]Recently M. Herman has proved Arnold's conjecture for a dense set of irrational rotation numbers. Using this result our condition (b) may be eliminated.

Let x be a fixed point of a diffeomorphism $g \in \text{Diff}(M)$. Then x is called quasi hyperbolic if one of the following three conditions holds:

$-T_x g$ has one eigenvalue one, the other eigenvalues have norm different from 1 and there is a g-invariant curve α through x, tangent to the eigenvector of $T_x g$ with eigenvalue one, such that $g|\alpha$ has 2nd, but not 3rd, order contact with the identity; in this case x is also called a saddle-node;

$-T_x g$ has one eigenvalue -1, the other eigenvalues have norm different from 1 and there is a g-invariant curve α through x, tangent to the eigenvector of $T_x g$ with eigenvalue -1, such that $g^2|\alpha$ has 3rd, but not 4th, order contact with the identity; in this case x is called a flip;

$-T_x g$ has a pair $\lambda \neq \bar{\lambda}$ of eigenvalues on the unit circle, the other eigenvalues have norm different from 1 and there is a g-invariant surface α through x, tangent to the generalized eigenspace of the pair λ, $\bar{\lambda}$ such that the 3-jet makes $g|\alpha$ an attractor or repellor; in this case we call x a Hopf point (or orbit).

A periodic point x of g with period n is quasi hyperbolic, saddle-node, flip or Hopf, if it is such a fixed point for g^n. The stable and unstable sets of quasi hyperbolic periodic points are also manifolds; in the case of a saddle-node they have boundaries.

Let x be a saddle-node for $g \in \text{Diff}(M)$. Then there is a locally invariant foliation with smooth leaves, the strong stable foliation F^s, of the stable manifold of x which is uniquely characterized by the following:

—the boundary of $W^s(x, g)$ is a leaf of F^s;

—g maps leafs of F^s to leafs of F^s.

(For related foliations see [2].) The leaf of F^s through x is called the strong stable manifold of x. We say that x is s-critical if there is some hyperbolic periodic point y of g whose unstable manifold $W^u(y, g)$ has a nontransversal intersection with one of the leafs of F^s. Interchanging stable and unstable in the above definition we obtain the definition of u-critical. We say that x is semicritical if it is either s-critical or u-critical; x is called bicritical if it is both s-critical and u-critical.

We now consider some generic conditions on the arcs.

DEFINITION OF GENERIC ARCS. The set $\mathfrak{G} \subset \mathfrak{A}$ of generic arcs in $\mathfrak{A}$ is defined as follows.

An element $f \in \mathfrak{A}$ is in $\mathfrak{G}$ if

—for each $t \in [0, 1]$ f_t has at most one nonhyperbolic periodic point which is quasi hyperbolic and all stable, strong-stable, unstable and strong-unstable manifolds intersect transversally or all periodic points are hyperbolic and there is one orbit in which a stable and unstable manifold intersect nontransversally with 2nd order contact;

—the quasi hyperbolic periodic orbits and the nontransversal intersections unfold generically.

For the proof that $\mathfrak{G}$ is a residual subset of $\mathfrak{A}$, see [3], [4], [7].

THEOREM. *Suppose $f \in \mathfrak{G} \subset \mathfrak{A}$.*

(1) *f is left stable if and only if all stable and unstable manifolds of periodic orbits of f_b intersect transversally.*

(2) *f is stable if and only if f is left stable, the quasi hyperbolic periodic orbit is not a bicritical saddle-node or a Hopf orbit and f_b has no cycles;*

(3) *f is continuously stable if and only if f is stable and f_b has no semi-critical saddle-node.*

We remark in closing that some of the facts and methods involved in proving these results also apply to stability questions on arcs beginning at general Axiom A diffeomorphisms and flows.

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