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ON DOMAINS OF MAXIMAL MONOTONE OPERATORS

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1. **Introduction.** Let H be a real Hilbert space. The single-valued operator A mapping $D(A) \subseteq H$ into H is monotone if it satisfies $(A(u) - A(v), u - v) \geqslant 0$, $\forall u, v \in D(A)$. A is called maximal monotone if there does not exist any multi-valued monotone extension of A. Browder [2] has shown that for a monotone, hemicontinuous operator A there exists no monotone extension $\widetilde{A}$ satisfying $D(\widetilde{A}) = D(A)$. That means that the only possibility of extending the operator is to extend its domain. The purpose of this paper is to give a condition which guarantees that its domain is maximal relative to the operator.

Our result depends upon the concept of a weak domain-closed operator, which arises in our theory in a natural way. In the linear case we show that selfadjointness is equivalent to that weak domain-closed plus symmetry. Given a monotone, hemicontinuous, weak domain-closed operator A, we then show that a sufficient condition for its maximality is the existence of a linear, positive selfadjoint operator having the same domain.

2. **Weak domain-closed operators.**

DEFINITION. Let A be a nonlinear operator with dense domain $D(A)$. We call A weak domain-closed, when from $u_k \rightharpoonup u$ and $A(u_k) \xrightarrow{D(A)} w$ it follows that $u \in D(A)$. By $A(u_k) \xrightarrow{D(A)} w$ we mean that $(A(u_k), v) \longrightarrow (w, v), \forall v \in D(A)$.

The following lemma shows that the concept of a weak domain-closed operator is related to the maximality of its domain:

LEMMA. *A linear operator L with dense domain is selfadjoint if and only if it is symmetric and weak domain-closed.*

PROOF. Suppose that L is selfadjoint and $u_k \in D(L)$ is a sequence with $u_k \rightharpoonup u$ and $(Lu_k, v) \longrightarrow (w, v), \forall v \in D(L)$. Then it follows from $(Lu_k, v) = (u_k, Lv) \longrightarrow (u, Lv)$ that $(u, Lv) = (w, v)$, $\forall v \in D(L)$. Consequently, $u \in D(L)$ and L is weak domain-closed.

In order to prove the other direction we suppose for given $u, v \in H$ that the following equations hold: $(u, Lv) = (w, v)$, $\forall v \in D(L)$. Let $u_k \in D(L)$ be a sequence with $u_k \rightharpoonup u$. Then it follows that

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$$(Lu_k, v) = (u_k, Lv) \rightharpoonup (u, Lv) = (w, v), \quad \forall v \in D(L).$$

Since L is weak domain-closed, we conclude that $u \in D(L)$ and by symmetry $w = Lu$.

3. Principal result.

THEOREM. *Let A be a monotone, hemicontinuous operator and let L be a linear, positive, selfadjoint operator. Suppose that A is weak domain-closed and $D(A) = D(L)$. Then A is maximal monotone.*

PROOF. By a theorem of Minty (see [1]) it is sufficient for the maximality of A that, for any $y \in H$, the equation $y = A(u) + u$ has a solution $u \in D(A)$. Without loss of generality we can suppose that $y = 0$, since we can replace the operator $A(u)$ by the operator $\widetilde{A}(u) = A(u) - y$, for arbitrary $y \in H$. Replacing $\widetilde{A}(u)$ by $A(u)$ we consider the operators

$$F_k = (I + kL)^{-1} A (I + kL)^{-1}, \qquad k > 0.$$

These operators are monotone, hemicontinuous and defined on all of H. Thus they are maximal monotone. Therefore the following equations have solutions (see [1], [2]):

$$0 = F_k(h_k) + h_k$$

and

$$0 = A(u_k) + u_k + 2kLu_k + k^2L^2u_k \quad \text{with } u_k = (I + kL)^{-1}h_k.$$

From $\|u_k\|^2 \leqslant \|h_k\|^2 = -(F_k(h_k), h_k) \leqslant \|F_k(0)\| \, \|h_k\|$ it follows that the sequences u_k and h_k are both bounded and both converge to the same u in the weak topology for $k \longrightarrow 0$. If we take into account that $kLu_k \rightharpoonup 0$ it follows that $u_k \rightharpoonup u$ and $A(u_k) \xrightarrow{D(A)} -u$. Using the fact that A is weak domain-closed, we conclude that $u \in D(A)$.

For any $v \in D(A)$ we define $w_k = (I + kL)v$. Then it follows by means of the inequalities

$$\begin{aligned} 0 &\leqslant (F_k(h_k) + h_k - F_k(w_k) - w_k, h_k - w_k) \\ &= -(A(v), u_k - v) - ((I + kL)v, (I + kL)(u_k - v)), \end{aligned}$$

that $0 \leqslant -(A(v) + v, u - v), \forall v \in D(A)$.

Since A is hemicontinuous and since u belongs to $D(A)$, it follows that u is a solution of $0 = A(u) + u$, completing the proof.

4. Applications.

I. Let A be a monotone, hemicontinuous, bounded operator and let L be a linear, positive, selfadjoint operator satisfying the condition $D(L) \subset D(A)$. Then $A + L$ is maximal monotone.

PROOF. Given a sequence $u_k \in D(L)$ and given $u, v \in H$ such that $u_k \rightharpoonup u$ and $(A(u_k) + Lu_k, v) \longrightarrow (w, v)$, $\forall v \in D(L)$. The sequence $A(u_k)$ is bounded and therefore there exists a $w' \in H$ with $(Lu_k, v) \longrightarrow (w - w', v)$, $\forall v \in D(L)$. If we make use of the above lemma, it follows that $u \in D(L) = D(A + L)$. Hence $A + L$ is weak domain-closed and in view of the above theorem $A + L$ is maximal monotone.

II. Let A be an operator defined on a linear, dense domain which is weak domain-closed and for which the Gâteaux-derivatives A'_u satisfy the condition

$$\lim_{t \to 0} \| A'_{u+tv}h - A'_u h \| = 0, \quad \forall u, v, h \in D(A).$$

Then, if all derivatives are positive and symmetric and if one derivative is positive and selfadjoint, it follows that A is maximal cyclically monotone.

PROOF. The cyclical monotonicity means that $\forall n > 0$, $\forall u_i \in D(A)$,

$$(A(u_n), u_n - u_0) + \cdots + (A(u_1), u_1 - u_2) + (A(u_0), u_0 - u_1) \geqslant 0.$$

In [3] it is shown that an operator, whose derivatives satisfy the above continuity condition, is cyclically monotone if and only if all derivatives are positive and symmetric. Thus the statement follows.

REFERENCES

1. H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert,* North-Holland Math. Studies, no. 5, North-Holland, Amsterdam, 1973. MR 50 #1060.

2. F. E. Browder, *Problèmes non linéaires,* Séminaire de Mathématiques Supérieures, no. 15 (Été, 1965), Presses Univ. Montréal, Montreal, Que., 1966. MR 40 #3380.

3. J. Weyer, *Zyklische Monotonie eines nicht-linearen Operators und Symmetrie seiner Linearisierung,* Diplomarbeit Köln, 1974.

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