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PROBABILISTIC FOUNDATIONS OF QUANTUM THEORIES AND RUBIN-STONE SPACES

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ABSTRACT. A construction of Mielnik probability spaces of dimension 2 is given in terms of Rubin-Stone functionals, and conversely it is shown that if a certain subset of a real linear space is supplied with 2 dimensional Mielnik probability space structure then the space becomes a Rubin-Stone space (in final analysis, a generalized inner product space). Analogous results are given for generalized inner product spaces in the sense of Nagumo.

1. **Introduction.** In [1], H. Rubin and M. H. Stone define a class of generalized inner product spaces. In this note, we announce some results concerning the use of these spaces as concrete representation spaces for quantum states. Our approach is a probabilistic one and follows the ideas and methods in [2] and [3].

DEFINITION 1.1. A linear space N over the real, complex, or quaternionic number system is a *Rubin-Stone space* if the following four postulates are satisfied:

POSTULATE 1. On N there is defined a nonnegative real function q such that

$$q(x + y) + q(x - y) = 2q(x) + 2q(y).$$

POSTULATE 2. As a function of the real number α the quantity $q(\alpha x)$ is bounded in some neighborhood of $\alpha = 0$ for each x.

POSTULATE 3. In the complex and quaternionic cases the relations $q(x) = q(ix)$, $q(x) = q(ix) = q(jx) = q(kx)$ hold for the imaginary units i and i, j, k respectively.

POSTULATE 4. If $q(x) = 0$, then $x = \theta$.

DEFINITION 1.2. Let S be a nonempty set, and let p be a real-valued function defined on $S \times S$ such that

(A) $0 \leqslant p(a, b) \leqslant 1$ and $a = b \Longleftrightarrow p(a, b) = 1$,

(B) $p(a, b) = p(b, a)$ for all $a, b \in S$.

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[1] Portions of these results will appear in a doctoral thesis of S. J. Guccione, Jr. at the University of Missouri-Rolla.

Two elements a and b in S are orthogonal if $p(a, b) = 0$. A subset R of S is an orthogonal system if any two distinct elements of R are orthogonal. From Zorn's lemma it follows that there exists a maximal orthogonal system B which is called a basis in S.

Let B be a basis and F_B be the class of all finite subsets F of B. Define $p(a, F) = \Sigma_{b \in F}\, p(a, b)$ for all $a \in S$. Then the following property of (S, p) is also postulated:

(C) For each basis B and for each $a \in S$, $\sup_{F \in F_B} p(a, F) = 1$.

Any pair (S, p) satisfying axioms (A), (B), (C), is called a probability space.

Let B_1 and B_2 be two bases in S; then, as shown in [2], B_1 and B_2 have the same cardinal number. This cardinal number is called the dimension of (S, p).

2. **Results.** In the way of relationships between Rubin-Stone spaces and probability spaces, we have the following two theorems.

THEOREM 1. *Let N be a topological linear space with a Rubin-Stone functional q. Then $(N - \{\theta\}, p)$ is a probability space of dimension 2, where*

$$p(x, y) = \frac{q(x + y)}{2[q(x) + q(y)]}.$$

THEOREM 2. *Let N be a real linear space and let q: $N \longrightarrow [0, \infty)$ be such that $q(x) = q(-x)$ and $q(\alpha x + y)$ is a measurable function of α for each pair x, $y \in N$.*

If $S = N - \{\theta\}$ is a two-dimensional probability space with all bases of the form $B = \{b, -b\}$, $y \in S$, with $p(x, y) = q(x + y)/2[q(x) + q(y)]$, then N is a Rubin-Stone space.

3. **Remarks.** In [4], a statistical characterization of a certain class of generalized inner product spaces is given.

DEFINITION 3.1. A normed linear space N over the real or complex number field is a *Nagumo space* if the following condition is satisfied:

$(*)$ The center of gravity $\bar{x} = 1/n\Sigma_{k=1}^{n} x_k$ of an arbitrary set of points $(x_1, \ldots, x_n)$ minimizes the function $F(z) = \Sigma_{k=1}^{n} f(\|x_k - z\|)$, where $f(u)$ is a continuous real-valued function defined for $u \geqslant 0$, with $f(0) = 0, f(1) = 1$.

In [3], a weaker axiom (A^*) is formulated for probability spaces, namely,

$$(\mathrm{A}^*) \qquad 0 \leqslant p^*(a, b) \leqslant 1 \text{ and } p^*(a, b) = 1 \Rightarrow a = b.$$

Using modified probability spaces, i.e. a pair (S, p^*) satisfying axioms (A^*), (B), (C), we have the following result:

THEOREM 3. *Let N be a Nagumo space and let*

$$F^* = \{f | f \in C_R[0, \infty),\ f(x) = 0 \Longleftrightarrow x = 0,$$
$$f(1) = 1, f \geqslant 0 \text{ and } f \text{ satisfies } *\}.$$

Then $S = N - \{\theta\}$ *is a modified probability space of dimension* 2, *where*

$$p(x, y) = \frac{f(\|x + y\|)}{2[f(\|x\|) + f(\|y\|)]}.$$

The proofs and details of the above theorems will appear elsewhere.

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