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Self-adjoint operators, by William G. Faris, Lecture Notes in Mathematics, No. 433, Springer-Verlag, Berlin, Heidelberg, New York, 1975, vi+115 pp., $7.80.

Quantum physics has greatly influenced the theory of self-adjoint operators throughout its development and continues to do so today. One problem arising in quantum physics, which is the main problem dealt with in the book under review, is the addition problem: When is the sum of two unbounded self-adjoint operators self-adjoint? More precisely, let A, B with domains $D(A)$, $D(B)$ be self-adjoint operators on a complex Hilbert space H. If the closure C of $A+B$ (defined on $D(A)\cap D(B)$) is self-adjoint, then we can regard C as "the" self-adjoint sum of A and B. More interesting are the cases in which C has many self-adjoint extensions, and the problem is to find the "right" one (if indeed there is a right one).

In quantum mechanics, kinetic and potential energy are described by self-adjoint operators, A, B say. Their sum is the total energy operator C, and to do quantum mechanics one must compute functions of it. One can do this (by the spectral theorem and the associated functional calculus) when C is self-adjoint. In particular, when C is self-adjoint, the dynamics of the system is described by the one parameter unitary group $\{\exp(-itC): t \in \mathbf{R}\}$, which is well defined. An example is the case of a spinless nonrelativistic quantum mechanical particle in a given potential. The Hilbert space is $H=L^2(\mathbf{R}^n)$ and the kinetic and potential energy operators are $A=-\Delta=-\sum_{j=1}^n \partial^2/\partial x_j^2$, $B=$the operator of multiplication by $V(x):\mathbf{R}^n \to \mathbf{R}$ ($B=V(x)$ for short). This set-up also describes two-body problems with no external potentials. The problem is to find the most general conditions on V so that $A+B$ (suitably interpreted) is self-adjoint.

One approach to the addition problem is via the Lie-Trotter product formula. For A, B self-adjoint and t real, let

$$U_n(t)=\{\exp(-itA/n)\exp(-itB/n)\}^n.$$

If the closure C of $A+B$ (on $D(A)\cap D(B)$) is self-adjoint then

$$\exp(-itC)=\text{strong}\lim_{n\to\infty} U_n(t).$$

On the other hand, if $U_n(t)$ has a strong limit which is a unitary group $\{\exp(-itC): t\in\mathbf{R}\}$, then we may *define* C to be the (Lie) sum of A and B. C is a certain self-adjoint extension of $A+B$ on $D(A)\cap D(B)$; for some purposes it can be regarded as the "right" one. This approach is developed nicely by P. R. Chernoff [**1**], but it is not discussed in Faris' book.

The approach favored by Faris is the one based on sesquilinear forms. If A is a self-adjoint operator, it determines a sesquilinear form S_A by the formula

$$(*) \qquad S_A(f, g)=\langle Af, g\rangle,$$

$\langle\cdot,\cdot\rangle$ denoting the underlying inner product. S_A has a natural extension to $\tilde{Q}(A)\equiv Q(A)\times Q(A)$ where $Q(A)=D(|A|^{1/2})$. Thus, supposing for instance that A is positive ($A\geqq 0$), we have

$$S_A(f, g)=\langle Af, g\rangle=\langle A^{1/2}f, A^{1/2}g\rangle;$$

this last expression is well defined for $f, g\in Q(A)$. Conversely, with certain sesquilinear forms S one can associate a self-adjoint operator so that $S=S_A$ holds (see $(*)$).

Now let A, B be two self-adjoint operators on H. Consider the associated sesquilinear form $S=S_A+S_B$ defined on $\tilde{Q}(A)\cap\tilde{Q}(B)$. If $S=S_C$ for a self-adjoint C, we define C to be the *form sum* of A and B. This definition is very natural in terms of quantum mechanics. A self-adjoint operator A represents an observable of a quantum mechanical system. For a unit vector $f\in H$, $\langle Af, f\rangle$ represents the expectation value of the observable A in the state f. This quadratic form (by polarization) determines S_A on $\tilde{Q}(A)$. In case A [resp. B] represents kinetic [resp. potential] energy, the total energy C should be determined by adding expectation values:

$$\langle Cf, f\rangle=\langle Af, f\rangle+\langle Bf, f\rangle,$$

and this can be done for $f\in Q(A)\cap Q(B)$ (which is often significantly larger than $D(A)\cap D(B)$). This definition has proved to be particularly useful when A and B are positive (or more generally when A is positive and the negative part of B is small relative to A in a suitable sense).

Faris has collected a substantial amount of recent research in his short volume, "recent" meaning since the appearance of T. Kato's book [**3**] in 1966. Faris' book is divided into four parts.

Part I begins with a review of the theory of self-adjoint operators through a statement of the spectral theorem, which is not proved. A bijective correspondence is established between positive self-adjoint operators and closed positive sesquilinear (or quadratic) forms. Various conditions are given ensuring that the form sum of A and B is self-adjoint. The applications include C. Friedman's results on Schrödinger operators $(-\Delta+V(x))$ with potentials having small support.

In part II the Hilbert space is taken to be L^2 of a measure space. The notion of positivity preserving ($f\geqq 0$ a.e. implies $Lf\geqq 0$ a.e.) is introduced. This leads to the notion of hypercontractive semigroups. Estimates of the form $A^2+B^2\leqq\text{const}(A+B)^2+\text{const}$ are established; these estimates are useful in quantum field theory. The Perron-Frobenius theorem is used to establish that in certain cases the infimum of the spectrum of the positive operator $A+B$ is an eigenvalue of multiplicity one. Among the applications is the following beautiful theorem of T. Kato: If $0\leqq V\in L^2_{\text{loc}}(\mathbf{R}^n)$, then the closure of $-\Delta+V$ on the $C^\infty(\mathbf{R}^n)$ functions with compact support is self-adjoint (on $L^2(\mathbf{R}^n)$).

Parts III and IV are short. Part III gives a slick treatment of the construction of semibounded self-adjoint extensions of a semibounded symmetric operator. These are parametrized by closed positive forms. Part IV deals with the relation between self-adjointness and the determination of a measure by its moments. It includes E. Nelson's theorem on analytic vectors and self-adjointness.

A notable feature of the book is that it is filled with interesting examples and counterexamples, the main recurring one being $H=L^2(\mathbf{R}^n)$, $A=-\Delta$, $B=V(x)$. There are also short introductions to some of the ideas of quantum mechanics and quantum field theory.

There are a few typographical errors. The author occasionally uses the terms "form" and "operator" interchangeably. The reviewer was unable to verify the application of the Sobolev inequalities on pp. 28, 32 using the form of the inequalities given in E. Stein's book, to which the author refers. Better references would be A. Friedman's book [**2**] and L. Nirenberg's paper [**4**]. In his neat presentation of the Heinz inequality (p. 30) the author fails to mention E. Heinz's name. But these criticisms are minor.

The author says: "In order to follow [this book] it should be sufficient to know real analysis and have some acquaintance with Hilbert space." This seems overly optimistic. The reader should have at least a moderately competent working knowledge of the spectral theorem. The book is well written. In some places the author is perhaps a bit stingy with details, but this should not deter any serious reader.

All in all, this reviewer found Faris' book to be a valuable and interesting survey of and introduction to some recent research in operator theory. People interested in the interplay between operator theory and quantum physics should also look at the forthcoming important book of M. Reed and B. Simon [**5**].

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Scattering theory for the d'Alembert equation in exterior domains, by Calvin H. Wilcox, Lecture Notes in Mathematics, No. 442, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 184 pp., \$8.60.

The subject of this book is a mathematical model for the propagation of sound around obstacles. The basic problem is to describe the behavior of sound waves which impinge on an infinitely hard object occupying a compact region $\Gamma\subset\mathbf{R}^3$ (the analysis is carried out for $\Gamma\subset\mathbf{R}^n$). Roughly, one has an incoming wave u_- which is unaffected by the obstacle. The sound then reaches Γ where it is reflected, diffracted, and is generally subject to complicated physical processes. Eventually the intensity of sound near Γ dies out indicating that the sound wave has traveled away from Γ (the model is conservative so the only way for sound to disappear in one place is for it to appear somewhere else). Thus for large time one expects to find a wave u_+ which is unaffected by the obstacle.

The mathematical model is the following. The sound wave is described by a function $u:\mathbf{R}\times(\mathbf{R}^3\backslash\Gamma)\to\mathbf{R}$ where $\partial u(t, x)/\partial t$ represents the difference between the pressure at place x and time t and the equilibrium pressure. With an appropriate choice of units the equation of motion for u is the wave (or d'Alembert) equation,

$$u_{tt}-\Delta u=0 \tag{1}$$

where $\Delta=\sum_{i=1}^n (\partial/\partial x_i)^2$. The Neumann boundary condition

$$\mathbf{n}\cdot \operatorname{grad}_x u=0 \quad \text{on } \mathbf{R}\times\partial\Gamma \tag{2}$$

($\mathbf{n}$=normal to $\partial\Gamma$) describes the interaction of sound with an infinitely hard obstacle. Waves in the absence of obstacles satisfy d'Alembert's equation on the entire space $\mathbf{R}\times\mathbf{R}^n$.

The intuition described above suggests that if u is a solution of (1), (2), then on any bounded set $\beta\subset\mathbf{R}^n$, $u(t)|_\beta\to 0$ as $t\to\infty$ and that there is a solution, u_+, of the wave equation on $\mathbf{R}\times\mathbf{R}^n$ with $u\approx u_+$ for t large. Similar assertions hold for $t\to-\infty$ with an associated free wave u_-. From a practical perspective one often knows the initial free wave u_- which then interacts with Γ then tends to u_+ as $t\to\infty$. The map $u_-\to u_+$ is called the scattering operator.