BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, 1976

STABILITY PROPERTIES OF THE CLASS OF ASYMPTOTIC MARTINGALES

BY ALEXANDRA BELLOW[1]

Communicated by Alexandra Bellow, December 18, 1975

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $N = \{1, 2, 3, \ldots\}$ and let $(\mathcal{F}_n)_{n \in N}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$, i.e. if $n \leqslant m$ then $\mathcal{F}_n \subset \mathcal{F}_n$. A *bounded stopping time* (with respect to the sequence $(\mathcal{F}_n)_{n \in N}$) is a mapping $\tau: \Omega \longrightarrow N$ such that $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in N$ and τ assumes only finitely many values. Let T be the set of all bounded stopping times. With the definition $\tau \leqslant \sigma$ if $\tau(\omega) \leqslant \sigma(\omega)$ for all $\omega \in \Omega$, T is a directed set "filtering to the right" (note that if $\tau \in T$, $\sigma \in T$, then $\tau \vee \sigma \in T$, $\tau \wedge \sigma \in T$). For $\tau \in T$ recall that $\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\tau = n\} \in \mathcal{F}_n \text{ for all } n \in N\}$ and that $\tau \leqslant \sigma$ implies $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$.

Let E be a *Banach space*. Let $X_n: \Omega \longrightarrow E$ for each $n \in N$. The sequence $(X_n)_{n \in N}$ is called *adapted* if $X_n: \Omega \longrightarrow E$ is Bochner $\mathcal{F}_n$-measurable for each $n \in N$.

The notion of *asymptotic martingale* has received a great deal of attention in the last few years; it provides a unified and elegant treatment for martingales, submartingales, supermartingales, quasimartingales [1], [2], [5]. We recall its definition:

DEFINITION. *An adapted sequence $(X_n)_{n \in N}$ of E-valued random variables is called an E-valued asymptotic martingale if X_n is Bochner integrable,* i.e. $\int \|X_n(\omega)\| \, dP(\omega) < \infty$ *for all $n \in N$ and $(\int X_\tau)_{\tau \in T}$ converges in the norm topology of E.*

We recall the fundamental a.e. convergence theorems for asymptotic martingales:

(I) Let $(X_n)_{n \in N}$ be a real-valued asymptotic martingale. Suppose that $\sup_{n \in N} \int |X_n| < \infty$. Then $(X_n)_{n \in N}$ converges to a limit a.e. (see [1]).

(II) Let $(X_n)_{n \in N}$ be an E-valued asymptotic martingale. Suppose that $\sup_{\tau \in T} \int \|X_\tau\| < \infty$. Then even under the best circumstances (if E is Hilbert space l^2) the sequence $(X_n)_{n \in N}$ need not converge in the norm topology of E, but only weakly a.e. (see [2]).

Nevertheless the following is true without any restriction on the Banach space E (see also [2], Lemma 2):

AMS (MOS) subject classifications (1970). Primary 60G40, 60G45; Secondary 28A65, 28A20.

[1]The main results of this note were announced in a Colloquium lecture given at the Hebrew University of Jerusalem, on November 27, 1975.

THEOREM 1. *Let E be a Banach space. Let $(X_n)_{n \in N}$ be an E-valued asymptotic martingale. For each $\tau \in T$ let $\mu_\tau(A) = \int_A X_\tau$, for $A \in \mathcal{F}_\tau$. Then:*

(1) *The family $(\mu_\tau(A))_\tau$ converges to a limit, $\mu(A)$, for each $A \in \mathcal{F}_\infty = \bigcup_{\tau \in T} \mathcal{F}_\tau = \bigcup_{n \in N} \mathcal{F}_n$, and the convergence is "uniform" on $\mathcal{F}_\infty$, that is for each $\epsilon > 0$ there is $\tau_0 \in T$ such that*

$$\sigma \in T,\ \sigma \geqslant \tau_0 \Rightarrow \|\mu_\sigma(A) - \mu(A)\| \leqslant \epsilon \quad \text{for all } A \in \mathcal{F}_\sigma.$$

(2) *Furthermore if $\sup_{n \in N} \int \|X_n\| < \infty$, then there is $M > 0$ such that $\|\mu_\tau(A)\| \leqslant M$ for each $\tau \in T$ and $A \in \mathcal{F}_\tau$.*

2. In the Lemma that follows we assume that: Ω is a set, $\mathcal{A}$ a Boolean algebra of subsets of Ω, T a directed set filtering to the right for "$\leqslant$", and $(\mathcal{A}_t)_{t \in T}$ an increasing family of sub-algebras of $\mathcal{A}$, that is: $s \leqslant t \Rightarrow \mathcal{A}_s \subset \mathcal{A}_t$.

For any real-valued bounded additive set function v defined on a Boolean algebra of subsets of Ω, we write $v = v^+ - v^-$ for the Jordan decomposition of v (see [4, pp. 98–99]).

The following Lemma may be regarded as a variant of E. H. Moore's double limit lemma (see [4, p. 28]):

LEMMA. *For each $t \in T$ let μ_t: $\mathcal{A}_t \longrightarrow R$ be an additive set function. We assume that:*

(i) *There is $M > 0$ such that $|\mu_t(A)| \leqslant M$ for each $t \in T$ and $A \in \mathcal{A}_t$.*

(ii) *The family $(\mu_t(A))_t$ converges to a limit, $\mu(A)$, for each $A \in \mathcal{A}_\infty = \bigcup_{t \in T} \mathcal{A}_t$, and the convergence is "uniform" on $\mathcal{A}_\infty$, that is for each $\epsilon > 0$ there is $t_0 \in T$ such that*

$$s \in T,\ s \geqslant t_0 \Rightarrow |\mu_s(A) - \mu(A)| \leqslant \epsilon \quad \text{for all } A \in \mathcal{A}_s.$$

Then $\lim_{t \in T} \mu_t^+(\Omega)$ and $\lim_{t \in T} \mu_t^-(\Omega)$ exist and equal $\mu^+(\Omega)$ and $\mu^-(\Omega)$, respectively.

With the notation of §1, the following result, first proved in [1] (see also [5]), is an easy consequence of Theorem 1 and the previous Lemma:

COROLLARY. *Let $(X_n)_{n \in N}$ be a real-valued asymptotic martingale and suppose that $\sup_{n \in N} \int |X_n| < \infty$. Then $(X_n^+)_{n \in N}$, $(X_n^-)_{n \in N}$ are asymptotic martingales.*

Before stating the next theorem we note that the class of continuous functions Φ: $R \longrightarrow R$ for which $\lim_{x \to +\infty}(\Phi(x)/x)$, $\lim_{x \to -\infty}(\Phi(x)/x)$ exist (finite or infinite) is quite large: it includes the piecewise linear functions, the convex functions, the concave function, the subadditive functions.

THEOREM 2. *Let Φ: $R \longrightarrow R$ be continuous and such that $\lim_{x \to +\infty}(\Phi(x)/x)$ and $\lim_{x \to -\infty}(\Phi(x)/x)$ exist and are finite. Let $(X_n)_{n \in N}$ be any real-valued asymp-*

totic martingale such that $\sup_{n\in N} \int |X_n| < \infty$. *Then* $(\Phi(X_n))_{n\in N}$ *is an asymptotic martingale and* $\sup_{n\in N} \int |\Phi(X_n)| < \infty$.

3. For simplicity we assume in this section that $\Omega = [0, 1]$, $\mathcal{F}$ = the σ-algebra of Borel sets and P a nonatomic probability measure. When the sequence of σ-algebras is not explicitly mentioned, it is assumed that $(\mathcal{F}_n)_{n\in N}$ is the "minimal", sequence, that is $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ for each $n \in N$.

In a certain sense Theorem 2 is best possible, as the following result shows:

THEOREM 3. *Let* $\Phi: R \longrightarrow R$ *be continuous and such that* $\lim_{x\to+\infty}(\Phi(x)/x) = +\infty$. *Then there is a real-valued asymptotic martingale* $(X_n)_{n\in N}$ *such that* $\sup_{n\in N} \int |X_n| < \infty$, $\sup_{n\in N} \int |\Phi(X_n)| < \infty$, *but* $(\Phi(X_n))_{n\in N}$ *is not an asymptotic martingale.*

REMARK. The standard examples of functions $\Phi: R \longrightarrow R$ satisfying $\lim_{x\to+\infty}(\Phi(x)/x) = +\infty$ are $|x|\log^+|x|$ and $|x|^p$ $(p > 1)$. The classical theorems from martingale theory concerning these functions [3, pp. 295–296] do not not carry over to asymptotic martingales, as Theorem 3 shows.

THEOREM 4. *Let* $S: \Omega \longrightarrow \Omega$ *be an ergodic measure-preserving transformation. There are then functions* $f \in L^1_+$ *such that if we set*

$$X_n = (f + f \circ S + \cdots + f \circ S^{n-1})/n,$$

for each $n \in N$, *then* $(X_n)_{n\in N}$ *is not an asymptotic martingale.*

To conclude: the notion of asymptotic martingale is an important and useful concept. Nevertheless it has its limitations, as Theorems 3 and 4 above show.

REFERENCES

1. D. G. Austin, G. A. Edgar and A. Ionescu Tulcea, *Pointwise convergence in terms of expectations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 30 (1974), 17–26.

2. R. V. Chacon and L. Sucheston, *On convergence of vector-valued asymptotic martingales*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 33 (1975), 55–59.

3. J. L. Doob, *Stochastic processes*, Wiley, New York; Chapman & Hall, London, 1953. MR 15, 445.

4. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

5. G. A. Edgar and L. Sucheston, *Amarts, a class of asymptotic martingales* (*Discrete parameters*), J. Multivariate Anal. (to appear).

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201