BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

THREE PARTIAL ORDERS ARISING FROM MULTIPLICATION ALTERATION BY TWO-COCYCLES

BY DAVE RIFFELMACHER

Communicated by James Bramble, December 8, 1975

1. **Introduction.** We introduce three partial orders arising from multiplication alteration by two-cocycles and show how some order properties of an algebra are related to its structure. Throughout this note C is an (associative) algebra with unit 1 over a commutative ring k, $\sigma = \Sigma a_i \otimes b_i \otimes c_i$ in $C \otimes C \otimes C$ is a C-two-cocycle with unity element e_σ, and C^σ is the k-algebra obtained from C and σ with product $x^\sigma * y^\sigma = (\Sigma a_i x b_i y c_i)^\sigma$. The reader is referred to [2] for the basic theory of multiplication alteration by two-cocycles. The author extends thanks to Moss Sweedler for directing this research.

2. **Definitions.** In this section we define three partial orders on the class of k-algebras which are due to Sweedler.

DEFINITION 2.1. *C, D k-algebras. C Amitsur dominates D* if there is a C-two-cocycle σ with $D \cong C^\sigma$ as k-algebras. C is an *Amitsur atom* if C Amitsur dominates D implies $D \cong C$.

EXAMPLE. The k-algebra of two by two upper triangular matrices with entries in k is an Amitsur atom.

Given a C-two-cocycle σ and a C^σ-two-cocycle τ, writing out $(x^\sigma)^\tau * (y^\sigma)^\tau$ suggests a candidate for a C-two-cocycle γ with $(C^\sigma)^\tau \cong C^\gamma$ via $(x^\sigma)^\tau \longleftrightarrow x^\gamma$. Direct calculation shows this element is a C-two-cocycle.

PROPOSITION 2.2. *Amitsur dominance is a partial order on the class of k-algebras.*

We define a map φ_σ: $C^\sigma \otimes C^{\sigma o} \longrightarrow C \otimes C^o$ by $\varphi_\sigma(x^\sigma \otimes y^{\sigma o}) = \Sigma a_i a_j x b_j \otimes (c_j b_i y c_i)^o$, where C^o is the opposite algebra of C. A straightforward calculation using the C-two-cocycle relations for σ shows that φ_σ is a k-algebra map. Let the change of rings functor induced by φ_σ from the category $M(C)$ of C-bimodules to the category $M(C^\sigma)$ of C^σ-bimodules be denoted $(\ )^\sigma$.

DEFINITION 2.3. *C, D k-algebras. C Hochschild dominates D* if there is a C-two-cocycle σ with $D \cong C^\sigma$ as k-algebras and $(\ )^\sigma$ is dense. C is a *Hochschild atom* if $(\ )^\sigma$ is dense for all C-two-cocycles σ.

PROPOSITION 2.4. *Hochschild dominance is a partial order.*

AMS (MOS) subject classifications (1970). Primary 16A48, 18H15; Secondary 16A16.

Let $A(C)$ be the category of k-algebras over C. Define a functor $F^\sigma\colon A(C) \longrightarrow A(C^\sigma)$ by $F^\sigma(C \xrightarrow{f} D) = C^\sigma \xrightarrow{f^\sigma} D^{f(\sigma)}$, where $f^\sigma(a^\sigma) = f(a)^{f(\sigma)}$ and $f(\sigma) = f^{\otimes 3}(\sigma)$, and

$$F^\sigma\left(\begin{array}{ccc} D & \xrightarrow{h} & E \\ & {}_{f}\nwarrow \;\; \nearrow_{g} & \\ & C & \end{array}\right) = \begin{array}{ccc} D^{f(\sigma)} & \xrightarrow{\bar{h}} & E^{g(\sigma)} \\ & {}_{f^\sigma}\nwarrow \;\; \nearrow_{g^\sigma} & \\ & C^\sigma & \end{array}$$

with $\bar{h}(d^{f(\sigma)}) = h(d)^{g(\sigma)}$.

DEFINITION 2.5. *C, D k-algebras. C categorically dominates D* if there is a C-two-cocycle σ with $D \cong C^\sigma$ as k-algebras and $F^\sigma\colon A(C) \longrightarrow A(C^\sigma)$ is dense. C is a *categorical atom* if F^σ is dense for all C-two-cocycles σ.

PROPOSITION 2.6. *Categorical dominance is a partial order.*

3. **Three characterization theorems.** We present one theorem to indicate how each of the partial orders from §2 may be used to characterize a type of k-algebra. The first theorem provides the converse of a result of [2].

THEOREM 3.1. *k field. C k-algebra of k-dimension n. The following are equivalent*:

(a) *C is a central simple k-algebra.*

(b) *C Amitsur dominates all k-algebras of k-dimension n.*

(c) *C Amitsur dominates $k \oplus \overset{n}{\cdots} \oplus k$ and $k[x]/(x^n)$.*

(d) *C Amitsur dominates a k-separable algebra and a k-purely inseparable (cf. [3]) algebra.*

INDICATION OF PROOF. (a) implies (b) is [2, Theorem 6.1]. The implication (d) ⇒ (a) follows from the behavior of the center $Z(C)$ of C and the Jacobson radical $J(C)$ of C under multiplication alteration by two-cocycles.

Before stating the next two theorems, we recall a class of C-two-cocycles mentioned in [2].

EXAMPLE (WATERHOUSE). Let B be a k-separable subalgebra of C with separability idempotent e. Then $\sigma_B = e \otimes 1 + 1 \otimes e - (1 \otimes e)(e \otimes 1)$ is a C-two-cocycle with $e_{\sigma_B} = 1$.

THEOREM 3.2. *k field. C an algebraic k-algebra with nilpotent Jacobson radical $J(C)$ and $C/J(C)$ locally finite. The following are equivalent*:

(a) *$(\ )^\sigma\colon M(C) \longrightarrow M(C^\sigma)$ is an equivalence for all σ.*

(b) *C is a Hochschild atom.*

(c) *All k-separable subalgebras of C are central.*

(d) *φ_σ is an isomorphism for all C-two-cocycles σ.*

INDICATION OF PROOF. The implication (b) ⇒ (c) follows from a study of the functors $(\ )^{\sigma_B}$ for Waterhouse two-cocycles. To show (c) implies (d), one proves that under the condition of (c) the hypothesis of the following lemma holds for all σ.

LEMMA 3.3. *Let* $\sigma = \Sigma a_i \otimes b_i \otimes c_i$ *be a C-two-cocycle and*

$$z_\sigma = \sum a_i a_j \otimes_{Z(C)} b_j^{\mathrm{o}} \otimes c_j b_i \otimes_{Z(C)} c_i^{\mathrm{o}}$$

in $C \otimes_{Z(C)} C^{\mathrm{o}} \otimes C \otimes_{Z(C)} C^{\mathrm{o}}$. *If* z_σ *is invertible,* φ_σ *is an isomorphism.*

THEOREM 3.4. *k field. C an algebraic k-algebra with nilpotent Jacobson radical J(C) and C/J(C) locally finite. The following are equivalent:*

(a) F^σ: $A(C) \longrightarrow A(C^\sigma)$ *is an equivalence for all* σ.

(b) *C is a categorical atom.*

(c) *C has no k-separable subalgebras (except k).*

(d) *All C-two-cocycles are invertible.*

INDICATION OF PROOF. The implication (b) ⇒ (c) follows from a study of the functors $F^{\sigma B}$ for Waterhouse two-cocycles. One proves (c) and (d) are equivalent using Wedderburn-Artin structure theory (cf. [1]) and the theory of purely inseparable algebras [3]. Then, after a reduction to the case $e_\sigma = 1$, one shows that under the hypotheses of (c) and (d), the condition of the following lemma holds for all C-two-cocycles σ with $e_\sigma = 1$.

LEMMA 3.5. *Let* $\sigma = \Sigma a_i \otimes b_i \otimes c_i$ *be a C-two-cocycle with* $e_\sigma = 1$ *and let*

$$\omega_\sigma = \sum (a_{i_1} a_{i_2} a_{i_3} a_{i_4})^{\mathrm{o}} \otimes b_{i_4} \otimes (c_{i_4} b_{i_3})^{\mathrm{o}} \otimes c_{i_3} b_{i_2} \otimes (c_{i_2} b_{i_1})^{\mathrm{o}} \otimes c_{i_1}$$

in $(C^{\mathrm{o}} \otimes C)^{\otimes 3}$. *If* ω_σ *is invertible, there is a* C^σ*-two-cocycle* τ *with* $F^\tau \circ F^\sigma = F^{1 \otimes 1 \otimes 1}$.

4. **Remark.** If δ is any element of $C \otimes C \otimes C$, we may still define a functor F^δ on $A(C)$. Then the image of F^δ is in $A(C^\delta)$ iff δ is a C-two-cocycle. This provides an easy proof of Proposition 2.2.

REFERENCES

1. A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloq. Publ., vol. 24, Amer. Math. Soc., Providence, R. I., 1939. MR 1, 99.

2. M. E. Sweedler, *Multiplication alteration by two-cocycles*, Illinois J. Math. **15** (1971), 302–323. MR **44** #5348.

3. ———, *Purely inseparable algebras*, J. Algebra 35 (1975), 342–355.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853