BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

MOMENTS OF MEASURES ON CONVEX BODIES

BY P. H. MASERICK

Communicated by Richard Goldberg, December 1, 1975.

ABSTRACT. A general notion of positive and bounded variation is introduced for functions on a commutative semigroup with involution. An integral representation for these functions is given. Applications to specific semigroups provide solutions to moment problems over convex bodies in R^n as well as a recovery of the Bochner-Herglotz-Weil theorem for discrete groups.

1. **Introduction.** The problem of moments can be viewed as one of identifying a class of functions on a semigroup with a class of measures. In this announcement we present integral representation theorems for linear functionals on algebras (Theorems 1 and 2) which enable us to solve moment problems for a wide class of compact sets. In particular if K is any compact convex subset of R^3 with nonvoid interior, necessary and sufficient conditions are given in §3 for a triple indexed sequence $f(n_1, n_2, n_3)$ to admit an integral representation of the form

$$f(n_1, n_2, n_3) = \int_K t_1^{n_1} t_2^{n_2} t_3^{n_3}\, d\mu(t) \qquad (t = (t_1, t_2, t_3)).$$

Here, of course, the semigroup, S, considered is all triples of nonnegative integers under coordinate addition. As in the case of Hausdorff's "little moment problems" (cf. [2] or [8]), the solution depends on certain linear combinations of shift operators. We consider the three coordinate shift operators E_1, E_2, E_3 defined on the functions f on S, e.g.,

$$(E_2 f)(n_1, n_2, n_3) = f(n_1, n_2 + 1, n_3).$$

If K is the simplex

$$\{t \in R^3 | t_1, t_2, t_3, 1 - t_1 - t_2 - t_3 \geqslant 0\},$$

then f admits a necessarily unique, nonnegative representing measure if and only if $(\Delta f)(0, 0, 0) \geqslant 0$, where Δ is any product of difference operators of the form $E_1, E_2, E_3, I - E_1 - E_2 - E_3$. A necessary and sufficient condition for the existence of a representing signed measure on the simplex is that

---

AMS (MOS) subject classifications (1970). Primary 44A50, 46A40; Secondary 46J25.

Key words and phrases. BV functions, positive definite functions, completely monotonic functions, integral representations, moment problems, variation, involution semigroups, involution algebras.

$$\lim_n \sum \binom{n}{i_1 i_2 i_3 i_4} |(I - E_1 - E_2 - E_3)^{i_4} f(i_1, i_2, i_3)| < \infty,$$

where $n = i_1 + i_2 + i_3 + i_4$. The simplex example is typical of all bounded nondegenerate convex polyhedra in that the difference operators used to describe those functions which admit representing measures are defined in terms of the facial functionals of the polyhedron K.

Let A be a real or complex commutative algebra with identity 1 and involution $*$. If A is real we assume $x^* = x$ for all $x \in A$. Let T be a subset of A such that:

(i) T is hermitian ($x^* = x$ for all $x \in T$);

(ii) $1 - T \subset \mathrm{Alg\,span}^+ T$ (for each $x \in T$ there exist products $x_1, \ldots, x_k$ of members of T and positive scalars $\alpha_1, \ldots, \alpha_k$ such that $x + \Sigma\alpha_i x_i = 1$);

(iii) $\mathrm{Alg\,span}\, T = A$ (for each $x \in A$ there exist scalars $\alpha_1, \ldots, \alpha_k$ and products $x_1, \ldots, x_k$ of members of T such that $x = \Sigma_{i=1}^k \alpha_i x_i$).

A linear functional f on A will be called *T-positive* if f is nonnegative on $\mathrm{Alg\,span}^+ T$. We let $P = \mathrm{Alg}^+ T$ and P' denote the set of T-positive linear functionals. Our main concern is one of characterizing P' and the span A' ($(P' - P') + i(P' - P')$ for the complex case). Solutions to classical moment problems as well as applications to positive-definite functions on groups, involution semigroups and positive functionals on B^*-algebras follow. The linear span of P' will be given the weak* topology and Γ will denote the set of all T-positive multiplicative linear functionals on A.

THEOREM 1. *A linear functional f is T-positive if and only if there exists a necessarily unique regular Borel measure μ_f such that*

$$f(x) = \int_\Gamma \chi(x)\, d\mu_f(\chi).$$

The map $\mu_f \longrightarrow f$ permits an obvious linear extension from the space $M(\Gamma)$ of all real or complex measures to the span P'. Let $|\mu|$ and $\|\mu\|$ denote the variation and total variation of a real or complex measure μ. The variation $|f|$ of $f \in A'$, can be defined by

$$|f|(x) = \int_\Gamma \chi(x)\, d|\mu_f|(\chi)$$

and the total variation by $|f|(1) = \|\mu_f\|$. The total variation will be described in terms of certain partitions of unity. Essential to these partitions are what we call "cycles". A *cycle* is a finite subset $\{x_i\}_{i=1,2,\ldots,k}$ of P such that $\{x_i\}_{i=1,2,\ldots,k-1} \subset T$ and $\Sigma_{i=1}^k x_i = 1$. For each $j = 1, 2, \ldots, k$ let $\{x_{ij}\}_i$ be a cycle and let $P_0 = \{\{x_{ij}\}_i\}_j$. Then for each integer n, the multinomial theorem implies that the collection

$$\left\{ \binom{n}{i_{11} \cdots} \binom{n}{i_{12} \cdots} \cdots \binom{n}{i_{lk} \cdots} \prod_j (x_{1j}^{i_1} x_{2j}^{i_2} \cdots) \,\middle|\, i_{1j} + i_{2j} + \cdots = n \right\}$$

is a partition of unity. Let $\mathcal{P}$ be a set of cycles such that each $x \in \mathcal{T}$ is a term of some cycle of $\mathcal{P}$. If $\mathcal{P}_0 \subset \mathcal{P}$ and f is a linear function then we define

$$\|f\|_{(\mathcal{P}_0,n)} = \sum \binom{n}{i_{11} \cdots} \binom{n}{i_{11} \cdots} \cdots \binom{n}{i_{1k} \cdots} \left| f\left[\prod_j (x_{1j}^{i_{1j}} x_{2j}^{i_{2j}} \cdots)\right] \right|$$

where it can be shown that $\|f\|_{(\mathcal{P}_0,n)}$ increases with $\mathcal{P}_0$ and n, and we define $\|f\| = \lim_{(\mathcal{P}_0,n)} \|f\|_{(\mathcal{P}_0,n)}$. When this limit is finite then f is said to be of $\mathcal{T}$-*bounded variation* (BV).

Our main theorem is as follows:

THEOREM 2. *A linear functional f is in the span of P' if and only if f is BV; a BV-functional being in the real span of P' if and only if $f(x^*) = f^*(x)$. Moreover $\|f\| = \|\mu_f\|$ if and only if f is BV.*

The motivation and main applications of Theorems 1 and 2 occur when $\mathcal{A}$ is an algebra of shift operators. For this let S be a commutative semigroup with identity 1 and involution * (cf. [3]). For each $x \in S$ define the shift operator E_x on the class of all complex valued functions f by $(E_x f)y = f(xy)$. Let $\mathcal{A}(E)$ denote the real or complex linear span of $\{E_x | x \in S\}$. Since $E_x E_y = E_{xy}$, if we define $E_x^* = E_{x^*}$ then $\mathcal{A}(E)$ is a commutative algebra with identity E_1. Let X be a subset of S such that every element of S is the product of finitely many (possibly repetitious) elements of elements of $X \cup X^*$. The linear functionals f on $\mathcal{A}$ can be identified with the functions f' on S via the map $f \longrightarrow f'$ where $f'(x) = f(E_x)$ for all $x \in S$, the semigroup semicharacters being identified with the multiplicative linear functionals. If $x = x^*$ and $\mathcal{T} = \{E_x, 1 - E_x | x \in X\}$, then the $\mathcal{T}$-positive and BV-functions correspond to the completely monotonic and BV-functions of [4]. Solvability of the moment problem on a 3-dimensional simplex, as mentioned in the introduction, is derived by taking X to be the unit vectors $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of the semigroup of all triples of integers with identity involution and $\mathcal{T} = \{E_1, E_2, E_3, 1 - E_1 - E_2 - E_3\}$. For an arbitrary involution semigroup, if we select $\mathcal{T} = \{1 + \sigma E_x/2 + \sigma^* E_x^*/2 | x \in X\}$ where $\sigma^4 = 1$, then the $\mathcal{T}$-positive functions correspond to the *-definite functions discussed in [3] and the BV-functions are identified with the BV-function defined in [5]. In case S is a group and $x^* = x^{-1}$, the $\mathcal{T}$-positive (or * definite) functions are merely the classically studied positive definite functions.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory,* Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

2. G. G. Lorentz, *Bernstein polynomials,* Math. Expositions, no. 8, Univ. of Toronto Press, Toronto, 1953. MR 15, 217.

3. R. J. Lindahl and P. H. Maserick, *Positive-definite functions on involution semigroups,* Duke Math. J. 38 (1971), 771–782. MR 45 #916.

4. P. H. Maserick, *Moment and BV-functions on commutative semigroups,* Trans. Amer. Math. Soc. 181 (1973), 61–75.

5. P. H. Maserick, *BV-functions, positive-definite functions and moment problems,* Trans. Amer. Math. Soc. 214 (1975), 137–152.

6. M. A. Naĭmark, *Normed rings,* 2nd rev. ed., "Nauka", Moscow, 1968; English transl., Wolters-Noordhoff, Groningen, 1970. MR 50 #8075; #8076.

7. R. R. Phelps, *Lectures on Choquet's theorem,* Van Nostrand, Princeton, N. J., 1966. MR 33 #1690.

8. D. V. Widder, *The Laplace transform,* Princeton Math. Ser., vol. 6, Princeton Univ. Press, Princeton, N. J., 1941. MR 3, 232.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802