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STABLE FACES OF A POLYTOPE

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The purpose of this announcement is to provide several new concepts for the study of the face-lattice of a polytope. The proofs of the results stated herein will appear elsewhere.

1. **Preliminaries.** Let P be a polytope with $\dim P \geqslant 0$ [4].

DEFINITION 1. A hyperplane S in aff P is said to be a *support* if P is entirely contained in one of the closed halfspaces determined by S. $\{F(P), \leqslant\}$ denotes the poset of faces of P which is indeed a lattice ($\wedge$: infimum, $\vee$: supremum; 1: greatest element; 0: least element). $A \subseteq F(P)$ is called an *order ideal* if $a \leqslant b$ and $b \in A$ implies $a \in A$. The ideal $[0, a]$ $(= \{b \in F(P) | 0 \leqslant b \leqslant a\})$ is called a *principal order ideal* (*generated by* a). A proper face a is called *split* [1] if $\{b | b \wedge a = 0\}$ is a principal order ideal and P is the direct convex sum of a and the generator of the primcipal order ideal. The faces 0 and 1 are called split.

A face of P is split if and only if it is a central element of the face-lattice.

DEFINITION 2. A pair of faces $a, b \in F(P) - \{1\}$ is said to be *orthogonal*, in symbols $a \perp b$, provided there exist two supports S_1, S_2 such that $a \subseteq S_1$, $b \subseteq S_2$ and $S_1 \cap S_2 = \varnothing$. We put: $a \perp 1$ $(1 \perp a)$ if and only if $a = 0$. If $A \subseteq F(P)$, define $A^\perp = \{b \in F(P) | b \perp a \text{ for all } a \in A\}$. If $\{a\}^\perp$ is a principal order ideal, then $a^\perp$ denotes its generator.

We have immediately: (i) $a \perp b \Rightarrow b \perp a$, (ii) $0 \perp a$ for all $a \in F(P)$, (iii) $a \perp a \Longleftrightarrow a = 0$, (iv) $a \perp b \Rightarrow a \wedge b = 0$, (v) $a \leqslant b$ and $b \perp c \Rightarrow a \perp c$.

2. **Mutually stabilizing pairs of faces.**

DEFINITION 3. A face $a \in F(P)$ is said to be *stable* provided there exists a face b such that a is a maximal element in the subposet $\{\{b\}^\perp, \leqslant\}$. $S(P)$ denotes the set of stable faces of P. A pair $a, b \in F(P)$ is called *mutually stabilizing* in symbols $a \perp\!\!\!\perp b$, if a is a maximal element in $\{\{b\}^\perp, \leqslant\}$ and b is a maximal element in $\{\{a\}^\perp, \leqslant\}$.

Note that facets and split faces are stable. One easily gets: (i) $a \perp\!\!\!\perp b \Rightarrow$

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$b \amalg a$, (ii) $a \amalg b \Rightarrow a \perp b$, (iii) $a \amalg b \Rightarrow a, b \in S(P)$, (iv) $0 \amalg a \Longleftrightarrow a = 1$, $1 \amalg a \Longleftrightarrow a = 0$.

THEOREM 1. *For every stable face a of a polytope there exists at least one face b such that the pair a, b is mutually stabilizing.*

THEOREM 2. *If* $a \amalg b$ $(a, b \neq 1)$ *then there exists exactly one pair of supports* S_1, S_2 *such that* $a \subseteq S_1$, $b \subseteq S_2$ *and* $S_1 \cap S_2 = \varnothing$.

THEOREM 3. $a \amalg b$ *if and only if* $a \perp b$ *and* $\dim(a \cup b) = \dim P$.

COROLLARY 4. $a \in F(P)$ *is a stable face if and only if there exists a face b such that* $a \perp b$ *and* $\dim(a \cup b) = \dim P$.

3. **Strong polytopes.**

DEFINITION 4. A polytope P is said to be *strong* provided (i) if $a \in S(P)$ then $\{a\}^{\perp}$ is a principal order ideal in $F(P)$, (ii) if $a \perp b$, $a \not\amalg b$ where $a, b \in S(P)$ then $\{a, b\}^{\perp\perp}$ is a principal order ideal in $F(P) - \{1\}$.

If P is a strong polytope then $a \amalg b \Longleftrightarrow a^{\perp} = b$ $(a \in S(P), b \in F(P))$, and therefore $S(P)$ is closed under the mapping $a \longrightarrow a^{\perp}$.

THEOREM 5. *If P is a strong polytope then* $\{S(P), \leqslant, {}^{\perp}\}$ *is an orthomodular poset. Its center coincides with the set of split faces.* $\{S(P), \leqslant, {}^{\perp}\}$ *is an orthomodular lattice if and only if* $\{a \vee b\}^{\perp}$ *is a principle order ideal in* $F(P)$ *for all* $a, b \in S(P)$.

(For definitions and basic properties of orthomodular posets see [3].)

A simplex is a strong polytope and $S(P) = F(P) = \{\text{split faces}\}$. The orthomodular poset $\{S(P), \leqslant, {}^{\perp}\}$ is Boolean (i.e.: a Boolean lattice).

THEOREM 6. *Let P be a strong polytope. If the orthomodular poset* $\{S(P), \leqslant, {}^{\perp}\}$ *is Boolean then P is a simplex.*

4. **States.** The issue of this section is to show that with each point oı a strong polytope P we can associate in a unique manner a state (probability measure) [2] on the orthomodular poset of stable faces. The treatment is purely geometrical.

Let P be a polytope with $\dim P \geqslant 1$, S_1, S_2 supports such that $S_1 \cap S_2 = \varnothing$ and $\nu \in \operatorname{aff} P$. Then there exist elements $\nu_1 \in S_1$, $\nu_2 \in S_2$ and a unique real number $\mu(\nu, S_1, S_2)$ such that

$$\nu = \mu(\nu, S_1, S_2)\nu_1 + (1 - \mu(\nu, S_1, S_2))\nu_2.$$

Now let P be a strong polytope ($\dim P \geqslant 0$). For all $\omega \in P$ and $a \in S(P)$ we define

$$\mu_\omega(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a = 1, \\ \mu(\omega, S_1, S_2) & \text{if } a \neq 0, 1, \end{cases}$$

where S_1, S_2 is the unique pair of supports such that $a \subseteq S_1$, $a^\perp \subseteq S_2$ and $S_1 \cap S_2 = \emptyset$ (see Theorem 2). Note that $0 \leqslant \mu_\omega(a) \leqslant 1$ for all $\omega \in P$ and $a \in S(P)$. Denote $\Delta = \{\mu_\omega | \omega \in P\}$.

THEOREM 7. *For every $\omega \in P$ the mapping $a \in S(P) \longrightarrow \mu_\omega(a) \in [0, 1]$ is a state on the orthomodular poset $\{S(P), \leqslant, {}^\perp\}$. The mapping $\omega \in P \longrightarrow \mu_\omega \in \Delta$ is one-to-one.*

THEOREM 8. *Let P be a strong polytope. Then*

(i) *Δ is a strongly order determining set of states for the orthomodular poset of stable faces;*

(ii) *μ_ω is a pure state (with respect to Δ) if and only if $\omega \in \mathrm{ext}\, P$;*

(iii) *μ_ω is a dispersion-free state if and only if, for all $a \in S(P)$, $\omega \notin a$ implies $\omega \in a^\perp$;*

(iv) *μ_ω is a superposition* [5] *of a family of states $\{\mu_{\omega_i} | i \in I\}$ if and only if ω belongs to the face generated by $\{\omega_i | i \in I\}$.*

5. **Remark.** In a subsequent paper we will give a characterization of those orthomodular posets that are ortho-order isomorphic to the orthomodular poset of stable faces of some strong polytope. The key notion in that investigation is a generalized version of the Jordan-Hahn decomposition of signed measures.

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