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ON THE TAMAGAWA NUMBER OF QUASI-SPLIT GROUPS

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1. **Introduction.** In this paper we give a formula for the Tamagawa number $\tau(G)$ (see [6]) of a connected semisimple quasi-split algebraic group G defined over an algebraic number field F. The method used is that of R. P. Langlands (see [2]).

Let $\mathbf{A}$ be the adeles of F; $G_{\mathbf{A}}$ the locally compact adele group of G in which the group G_F of F-rational points is embedded.

Let B be the Borel subgroup of G defined over F, and A the maximal torus of B defined over F. $\tau(A)$ is the Tamagawa number of A. L_F (resp. L_F^+) denotes the lattice of F-rational weights of G (resp. of the simply-connected form of G). Let c be the index $[L_F^+ : L_F]$. Then the main formula is

THEOREM. $\tau(G) = c\tau(A)$.

2. **Sketch of the proof.** Let P be the orthogonal projection of $L^2(G_F\backslash G_{\mathbf{A}})$ onto the space of constant functions. Langlands [2] observes the simple relation:

$$(1, 1)(P\varphi^{\sim}, P\psi^{\sim}) = (\varphi^{\sim}, 1)(1, \psi^{\sim}) \tag{1}$$

where $\varphi^{\sim}, \psi^{\sim} \in L^2(G_F\backslash G_{\mathbf{A}})$ and $(\cdot, \cdot)$ is the inner product on $L^2(G_F\backslash G_{\mathbf{A}})$. As

$$(1, 1) = \int_{G_F\backslash G_{\mathbf{A}}} dg, \tag{2}$$

the problem reduces to the computation of the remaining three terms in (1).

Let $G_\infty = \Pi_{v|\infty} G_{F_v}$ where F_v is the completion of F at the place v and "$v|\infty$" means that v is infinite. Let K_∞ be the maximal compact subgroup of G_∞, and $K_0 = \Pi_{v<\infty} G_{\mathcal{O}_v}$ where "$v < \infty$" means that v is finite, $\mathcal{O}_v$ is the maximal compact subring of F_v and $G_{\mathcal{O}_v}$ is the compact subgroup of G_{F_v} consisting of elements with coefficients in $\mathcal{O}_v$ and whose determinants are units. Put

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[1] This paper is based on the author's Ph. D. dissertation, written at Yale University under Professor G. D. Mostow. The problem and the approach were suggested by R. P. Langlands.

$K = K_\infty \cdot K_0$. Then there exists a finite set $\{g_i \in G_{\mathbf{A}} | 1 \leqslant i \leqslant n\}$ such that

$$G_{\mathbf{A}} = \bigcup_{i=1}^{n} B_{\mathbf{A}} g_i K. \tag{3}$$

Let N be the unipotent radical of B, pick continuous functions φ, ψ defined on $N_{\mathbf{A}} B_F \backslash G_{\mathbf{A}} / K$ such that we have a Fourier integral expression

$$\varphi(g) = \int_{|\lambda|=\lambda_0} \Phi^\lambda(g)\, d\lambda \tag{4}$$

for a suitable quasi-character λ_0 of $A_F \backslash A_{\mathbf{A}}$ and the series

$$\varphi^\sim(g) = \sum_{\gamma \in B_F \backslash G_F} \varphi(\gamma g) \tag{5}$$

converges to an element in $L^2(G_F \backslash G_{\mathbf{A}})$. Similarly, we have

$$\psi(g) = \int_{|\lambda|=\lambda_0} \Psi^\lambda(g)\, d\lambda.$$

The Φ, Ψ are functions in λ and g, and there exists a sesquilinear pairing $\langle \cdot , \cdot \rangle$ between these functions such that

$$(\varphi, 1) = \langle \Phi^\rho, 1 \rangle, \quad (1, \psi) = \langle 1, \Psi^\rho \rangle \tag{6}$$

where ρ is the half sum of the positive roots of G.

To evaluate the remaining terms $(\mathcal{P}\varphi, \mathcal{P}\psi)$, we introduce an unbounded self-adjoint operator $\mathcal{A}$ on the closed subspace $\mathcal{L}$ of $L^2(G_F \backslash G_{\mathbf{A}})$ generated by the functions $\varphi^\sim$ with φ of the form indicated above. If $\mathcal{E}(x)$ is a right continuous spectral resolution of $\mathcal{A}$, then we have

$$\mathcal{P} = \mathcal{E}((\rho, \rho)) - \mathcal{E}((\rho, \rho) - 0), \tag{7}$$

$$(\mathcal{P}\varphi^\sim, \mathcal{P}\psi^\sim) = \frac{1}{c\tau(A)} \lim_{s \to 1} \frac{\langle M(w, \rho^s)\Phi^{\rho^s}, \Psi^{w\rho^{-\bar{s}}} \rangle}{L(s, A)}, \tag{8}$$

where w is the element of the Weyl group that sends every positive root to negative root, s is a complex number, $L(s, A)$ is the L-function of A (see [4], [5]) and $M(w, \rho^s)$ is a linear map on a vector space of functions on $N_{\mathbf{A}} B_F \backslash G_{\mathbf{A}} / K$.

There exists a finite set S of places of F such that

$$M(w, \rho^s)\Phi^{\rho^s}(g) = \left(\prod_{v \notin S} \int_{N_{F_v}} \Phi^{\rho^s}(w n_v)\, dn_v \right) \left(\int_{N_S} \Phi^{\rho^s}(w n_S g_S)\, dn_S \right), \tag{9}$$

where $g = (g_v) \in G_{\mathbf{A}}$ is such that $g_v = 1$ if $v \notin S$, $n_S \in N_S = \Pi_{v \in S} N_{F_v}$.

Let $\bar{N}$ be the unipotent radical of the Borel subgroup opposite to B. Write $\bar{N}^w = w^{-1} N w \cap \bar{N}$. Then we have

$$\int_{\bar{N}^w_{F_v}} \Phi^\lambda(\bar{n})\, d\bar{n} = \frac{\det(I - |\varpi| \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}} w)}{\det(I - \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}} w)} \tag{10}$$

where $\Phi^\lambda(1) = 1$ (for notation see [3], [4]). Formula (10) is proved first for all rational rank one quasi-split groups by explicit computation and then for the general case by the method of Bhanu-Murti, Gindikin and Karpelevic [1]. From (10) we get

$$\text{(11)} \quad \lim_{s\to 1} \prod_{v\notin S} \int_{N_{F_v}} \Phi^{\rho^s}(wn_v)\, dn_v = \left(\lim_{s\to 1} \prod_{v\notin S} L_v(s, A)\right)\left(\prod_{v\notin S} \text{volume } G_{0_v}\right).$$

The remaining integral in (9) is calculated by comparing the decomposition of the measure on $G_{\mathbf{A}}$ according to the Iwasawa decomposition and the Bruhat decomposition. We get

$$\text{(12)} \qquad \lim_{s\to 1} \int_{N_S} \Phi^{\rho^s}(wn_S g_S)\, dn_S = \frac{\langle \Phi^\rho, 1\rangle \Pi_{v\in S} L_v(1, A)}{\Pi_{v\notin S} \text{ volume } G_{0_v}}.$$

The theorem now follows immediately from (1), (2), (6), (8)–(12).

It follows from our theorem that Weil's conjecture on Tamagawa is true for quasi-split group.

REFERENCES

1. S. G. Gindikin and F. I. Karpelevič, *Plancherel measure for Riemann symmetric spaces of nonpositive curvature,* Dokl. Akad. Nauk SSSR 145 (1962), 252–255 = Soviet Math. Dokl. 3 (1962), 962–965. MR 27 #240.

2. R. P. Langlands, *The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups,* Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R. I., 1966, pp. 143–148. MR 35 #4226.

3. ———, *Problems in the theory of automorphic forms,* Lectures in Modern Analysis and Applications, III, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin, 1970, pp. 18–61. MR 46 #1758.

4. K. F. Lai, *On the Tamagawa number of quasi-split groups,* Ph. D. Dissertation, Yale, 1974.

5. T. Ono, *Arithmetic of algebraic tori,* Ann. of Math. (2) 74 (1961), 101–139. MR 23 #A1640.

6. ———, *On Tamagawa numbers,* Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R. I., 1966, pp. 122–132. MR 35 #191.

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