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EXAMPLES OF ELLIPTIC COMPLEXES

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The main purpose of this note is to give natural geometric examples of elliptic complexes for which the Poincaré lemma fails. Indeed:

(a) There are natural (and even involutive) elliptic complexes which are not formally exact, and whose local cohomology is infinite (Examples 2, 3). On the other hand:

(b) An arbitrary locally exact elliptic complex need not be formally exact (cf. Example 4′).

These remarks reflect interestingly on the outstanding problem in the theory (Spencer's conjecture): Is a formally integrable formally exact elliptic complex locally exact? (See Goldschmidt [2] for a complete analysis of the formal theory.) Thus (a) demonstrates forcibly the *independence* of the hypotheses, whereas (b) shows that the hypothesis of formal exactness is not always *necessary*.

Most of our examples take the following form: Let E be a subbundle of $\Lambda^p(\mathbf{R}^{n*})$; let $\underline{E}$ denote the sheaf of germs of sections of E. Then there are complexes of the following types:

$$\text{(I)} \qquad \underline{\Lambda}^{p-2} \xrightarrow{d} \underline{\Lambda}^{p-1} \xrightarrow{\pi d} \underline{\Lambda^p/E};$$

$$\text{(II)} \qquad \underline{E} \xrightarrow{d|E} \underline{\Lambda}^{p+1} \xrightarrow{d} \underline{\Lambda}^{p+2}.$$

Note to begin with that the cohomology of (I) is equivalent to the space of closed sections of E, i.e., the solution space of a *homogeneous* system of equations. One of our basic observations is then:

(c) There are nontrivial examples of these types which are elliptic (cf. Examples 2, 3).

On the other hand, Spencer's conjecture itself cannot be disproved within the context of such examples: if E is nontrivial, (I) is not formally exact; if (II) is elliptic (no further hypotheses), one checks it is locally exact.

Constant coefficient examples.

EXAMPLE 1 (NIRENBERG). An arbitrary elliptic complex need not be formally or locally exact. Over $\mathbf{C}^n$ construct

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$$0 \longrightarrow \underline{\Lambda}^0 \oplus \underline{0} \xrightarrow{\bar\partial \oplus 0} \underline{\Lambda}^{0,1} \oplus \underline{\Lambda}^0 \xrightarrow{\bar\partial \oplus \bar\partial} \underline{\Lambda}^{0,2} \oplus \underline{\Lambda}^{0,1} \longrightarrow \cdots$$

The cohomology at $\underline{\Lambda}^{0,1} \oplus \underline{\Lambda}^0$ is infinite.

We will say that a complex $\underline{E} \xrightarrow{D_0} \underline{F} \xrightarrow{D_1} \underline{G}$ is "natural" if D_0 is induced by a *surjective* bundle map $\varphi_D\colon J^kE \longrightarrow F$. This formal condition precludes artificial constructions such as the above.

EXAMPLE 2. Let $\Lambda^2_\pm$ be the space of $*$-invariant (resp. anti-invariant) 2-forms on $\mathbf{R}^4$ (standard metric). Then

$$0 \longrightarrow \underline{\Lambda}^0 \xrightarrow{d} \underline{\Lambda}^1 \xrightarrow{\pi_+ d} \underline{\Lambda^2_+} \longrightarrow 0$$

is natural, elliptic, formally integrable, and involutive (cf. [3] and [4]), yet the cohomology at $\underline{\Lambda}^1$ is infinite. The dual complex is

$$0 \longrightarrow \underline{\Lambda^2_-} \xrightarrow{d|\Lambda^2_-} \underline{\Lambda}^3 \xrightarrow{d} \underline{\Lambda}^4 \longrightarrow 0$$

and is locally exact as marked above. These complexes were discovered independently by Nigel Hitchin.

EXAMPLE 3. In 2 complex variables

$$0 \longrightarrow \underline{\Lambda}^0 \xrightarrow{d} \underline{\Lambda}^1 \xrightarrow{\pi_{1,1} \circ d} \underline{\Lambda}^{1,1} \xrightarrow{\partial\bar\partial} \underline{\Lambda}^{2,2} \longrightarrow 0$$

is elliptic, but noninvolutive as reflected by the second order continuation $\partial\bar\partial$. The cohomology at $\underline{\Lambda}^1$ is again infinite, but zero otherwise. This is the dual of the well-known resolution of the sheaf of germs of pluriharmonic functions.

EXAMPLE 4. Let ω be a symplectic form on a 4-manifold M. Then $\wedge\,\omega\colon \Lambda^1 \longrightarrow \Lambda^3$ is an algebraic isomorphism, and

$$0 \longrightarrow \underline{\Lambda}^0 \xrightarrow{d} \underline{\Lambda}^1 \xrightarrow{\pi d} \underline{\Lambda}^2/\omega \xrightarrow{\pi d(\wedge\omega)^{-1} d} \underline{\Lambda}^2/\omega \xrightarrow{d} \underline{\Lambda}^3 \xrightarrow{d} \underline{\Lambda}^4 \longrightarrow 0$$

is elliptic, with local cohomology one dimensional at Λ^1 and exactness holding elsewhere.

One generalization of Example 2 is the following: let $F\colon \mathbf{R}^k \otimes \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be an orthogonal multiplication (symbol of the Dirac operator in k variables). Let $E \subseteq \mathbf{R}^n$ be any subspace, with $E^\perp$ its orthocomplement. Then there is an elliptic *Dirac complex*

$$0 \longrightarrow \underline{E} \xrightarrow{D} \underline{\mathbf{R}}^n \xrightarrow{D_\perp^*} \underline{E}^\perp \longrightarrow 0.$$

Here σ_D and $\sigma_{D_\perp}$ are induced by restricting F to E and $E^\perp$ respectively. When D is involutive, the exactness of a Dirac complex becomes equivalent to a combinatorial criterion, the connectedness of a certain finite graph. This uses Ehrenpreiss [1] on constant coefficient systems and Kuranishi [4] on involutive systems. Example (2) above is equivalent to the Dirac complex arising from quaternion multiplication $H \otimes H \longrightarrow H$, with $E = \text{Span}\ (1)$, $E = \text{Span}(i, j, k)$.

Variable coefficient examples.

EXAMPLE 2′. There is no metric on a closed oriented manifold M^4 such that the corresponding Λ^2_+-complex is locally exact. Otherwise by sheaf theory we would find $H^3(M, \mathbf{R}) = H^4(M, \mathbf{R}) = 0$.

EXAMPLE 3′. There *are* local perturbations of (3) such that the cohomology at Λ^1 is finite. This is equivalent to exhibiting perturbations of the *homogeneous* Cauchy-Riemann equations for holomorphic functions with finite solution space. However there is no elliptic continuation analogous to $\partial\bar{\partial}$.

EXAMPLE 4′. Perturbing the symplectic form ω to a nondegenerate form $\tilde{\omega}$ such that $d((\wedge\tilde{\omega})^{-1}d\tilde{\omega}) \neq 0$, the elliptic complex

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{\pi d} \Lambda^2/\tilde{\omega}$$

is locally exact, but not formally exact. This is a quite general phenomenon which is not special to elliptic complexes.

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