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ON THE NONTRIVIALITY OF SOME GROUP EXTENSIONS GIVEN BY GENERATORS AND RELATIONS

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Let G be any group, $F = F(x_1, x_2, \ldots, x_n)$ the free group on n generators. Consider the group presentation $H = G^*F/R_1, R_2, \ldots, R_n$, where each relation R_i is a product of conjugates, by elements of G, of elements of F:

$$R_i = \prod_v g(i, v) x_{\alpha(i,v)}^{r(i,v)} g^{-1}(i, v).$$

Then G injects into H, and we want to know when H is genuinely larger than G. A criterion will be framed in terms of the Fox matrix, E, of the presentation:

$$(E)_{i,j} = \sum_v r(i, v) g(i, v)$$

where the sum is over those v for which $\alpha(i, v) = j$. The assumption that $H = G$ implies by a formal argument (or use of Fox free differential calculus) that E is invertible matrix over $\mathbf{Z}(G)$; to avoid this trivial case, we will assume $E \in \mathrm{GL}(n, \mathbf{Z}(G))$.

E may equally well be written $E = \Sigma M(g)g$, each $M(g) \in M(n, \mathbf{Z})$. For any finite dimensional representation ρ of G we define $\rho(E) = \Sigma M(g) \otimes \rho(g)$. Note that $\det 1(E) = \pm 1$.

Let A be the subgroup of $\mathrm{GL}(n, \mathbf{R}(G))$ generated by squares and commutators. Our main result is

THEOREM 1. *Assume G finite,* $\det 1(E) = 1$, *and n odd. If $E \notin \bigcup_{g \in G} \{Ag\}$ then G injects properly into H.*

(The case n even can be reduced to the preceding by adding a free generator to H and a relation which kills it.)

The proof needs several preliminary considerations. Let L be a compact connected Lie group of rank m, dimension d, and $\mathfrak{L}$ its Lie algebra. Let φ be any homomorphism of G into L; Ad φ is then a representation of G on $\mathfrak{L}$. By $\varphi(R_i)$ we mean the relation R_i with elements of G therein occurring replaced by their images under φ. Consider the map $f: L^n \longrightarrow L^n$ given by $f_i(x_1, x_2, \ldots, x_n) = \varphi(R_i) \cdot x_i$. The identity element of L^n is a fix point of f; another fix point of f assures H is larger than G.

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Using the techniques in [2], [3], we prove

LEMMA 1. *The Lefschetz number of f is* $[\det 1(E)]^m = [\det 1(E)]^d$.

(The Lefschetz number is here adjusted in sign to be equal to the algebraic sum of the indices at the fixed points.)

Using local coordinates supplied by L, and material in [1], we prove

LEMMA 2. *The index of f at the identity is* sgn det Ad $\varphi(E)$, *necessarily the determinant is* $\neq 0$.

Representations of the form Ad φ are necessarily proper orthogonal. Let $\rho: G \longrightarrow SO(k)$, and θ the natural inclusion of $SO(k)$ in $SO(k+1)$. One finds that $\mathrm{Ad}(\theta\rho \oplus \rho) = 2\,\mathrm{Ad}\,\rho \oplus \rho$, so

$$\det \mathrm{Ad}(\theta\rho + \rho)(E) = \det^2 \mathrm{Ad}\,\rho(E) \cdot \det \rho(E).$$

Hence

THEOREM 2. $[\det 1(E)]^d \det \mathrm{Ad}\,\varphi(E) > 0$, *for all homomorphisms φ if and only if* $[\det 1(E)]^k \det \rho(E) > 0$ *for all proper orthogonal representations* $\rho: G \longrightarrow SO(k)$, *for all k.*

Note that either criterion above depends only on the equivalence class of E in the Whitehead group of G. We do not have an intrinsic Wh(G) characterization of such elements, but Theorem 1 is a step in this direction.

For the sequel G is a finite group. W.l.o.g. we may also suppose $\det 1(E) = 1$.

THEOREM 3. $\det \rho(E) > 0$ *for all real representations ρ if and only if E belongs to A, defined just before Theorem* 1.

This result follows by piecing together appropriate decompositions of E in all the irreducible representations of G.

By a straightforward combinatorial argument we show

LEMMA 3. *If* $\det \rho(E) > 0$ *for all proper orthogonal representations ρ, then there exists an element $h \in G$ such that* $\det \varphi(E) \det \varphi(h) > 0$ *for all real representations φ.*

In case n is odd $E \cdot h$ satisfies the hypothesis of Theorem 3, and Theorem 1 is completely proved.

Our results yield nothing if G is finite abelian. Here is an example for G the dihedral group of order 10; $g^5 = h^2 = hgh^{-1}g = 1$. $E = (-1 + g - g^2 + g^3 + g^4) + h(1 - 2g + g^2)$ is a unit in $\mathbf{Z}(G)$ with inverse $(1 - g - g^2 + g^3 + g^4) + h(-2g + g^3 + g^4)$. Also $\det 1(E) = 1$. Consider the representations ρ_1 and ρ_2 with $\rho_1(g) = 1$, $\rho_1(h) = -1$ and

$$\rho_2(g) = \begin{pmatrix} \cos(2\pi/5), & \sin(2\pi/5) \\ -\sin(2\pi/5), & \cos(2\pi/5) \end{pmatrix}, \quad \rho_2(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\rho_1 \oplus \rho_2$ is proper orthogonal. It is left as an exercise to the reader to show that $\det(\rho_1 \oplus \rho_2)(E) = -4\cos^2(2\pi/5)$.

For another application, let G be the (possibly infinite) group generated by a collection $s_1, s_2, \ldots, s_v$ of elements of $\mathrm{SO}(k)$. If $a_i \in \mathbf{Z}$, $\Sigma a_i = 1$, and $\det(\Sigma a_i s_i) < 0$, then any associated group H is larger than G. For our hypotheses fix the Lefschetz number at 1, while the index at the identity is -1. $\Sigma a_i s_i$ may not be a unit in $\mathbf{Z}(G)$, but this is sometimes difficult to ascertain.

It is pleasant to record the benefit of several useful conversations with I. Berstein and M. Cohen. Indeed, this work arose because Professor Cohen had reduced the solution of a problem in simple homotopy theory to the proof of a related conjecture about group presentations.

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