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COBORDISM OPERATIONS AND SINGULARITIES OF MAPS

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If f is a differentiable map of smooth manifolds, the critical set $\Sigma(f)$ is not a manifold, in general. However, there is a *canonical* resolution of the singularities of $\Sigma(f)$ (for generic f), due to I. Porteous [6]. This resolution can be used to give a geometric description of T. tom Dieck's Steenrod operations in unoriented cobordism [7]. This was suggested to me by Jack Morava, as a parallel to my discription of ordinary mod 2 Steenrod operations using branching cycles of maps of n-circuits [5].

1. **Singularities of vector bundle maps.** Let $\xi^n = (E \longrightarrow X)$ and $\eta^p = (F \longrightarrow X)$ be real vector bundles over the smooth manifold X (without boundary), and let $g: E \longrightarrow F$ be a vector bundle map. That is, g is smooth, and for each $x \in X$, g sends the fiber E_x to the fiber F_x by a linear map g_x. The critical set $\Sigma(g)$ is $\{x \in X, \operatorname{rank}(g_x) < \min(n, p)\}$. Let $P(\xi) = (P(E) \longrightarrow X)$ be the projectification of ξ, i.e. the bundle whose fiber over x is the set of one-dimensional subspaces of E_x. Set $\widetilde{\Sigma}(g) = \{l \in P(E), l \subset \text{kernel } (g)\}$. The projection $\widetilde{\Sigma}(g) \longrightarrow X$ is proper, and if $n \leqslant p$, its image is $\Sigma(g)$. (If $n > p$, its image is all of X.)

LEMMA. (a) *If $g: \xi^n \longrightarrow \eta^p$ is a generic vector bundle map* [4] *over the d-manifold X, $\widetilde{\Sigma}(g)$ is a $(d - i)$-manifold, where $i = p - n + 1$.*

(b) *If $h: \xi^n \longrightarrow \eta^p$ is another such map, $\widetilde{\Sigma}(h) \longrightarrow X$ is properly cobordant with $\widetilde{\Sigma}(g) \longrightarrow X$.*

This lemma is proved by considering the canonical bundle map G over $\operatorname{Hom}(\xi, \eta)$. A vector bundle map $g: \xi \longrightarrow \eta$ defines a section of $\operatorname{Hom}(\xi, \eta) \longrightarrow X$, and $\widetilde{\Sigma}(g) \longrightarrow X$ is the pull-back of $\widetilde{\Sigma}(G) \longrightarrow \operatorname{Hom}(\xi, \eta)$ by this section.

It follows from Quillen's geometric description of smooth unoriented cobordism theory N^* [3] that this construction defines a natural transformation $\sigma: K(X) \longrightarrow N^*(X)$. If $K(X)$ is defined as the set of all pairs (ξ, η) of bundles over X, modulo the relation $(\xi \oplus \zeta, \eta \oplus \zeta) \sim (\xi, \eta)$, σ is induced by $(\xi, \eta) \longmapsto \widetilde{\Sigma}(g)$, where $g: \xi \longrightarrow \eta$ is a generic map. A dual $\bar{\sigma}$ is defined by $\bar{\sigma}[\xi, \eta] = \sigma[\eta, \xi]$.

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σ determines a family of stable cobordism characteristic classes σ_i, $i \in \mathbf{Z}$, by setting $\sigma_i(\eta^p) = \sigma(\epsilon^n, \eta^p) \in N^i(X)$, where ϵ^n is the trivial n-bundle over X, $n = p - i + 1$. If $\bar{\xi}$ is a stable inverse for ξ, $\sigma_i(\bar{\xi}) = \bar{\sigma}_i(\xi)$.

2. **Steenrod operations.** Thom's definition of characteristic classes gives a bijection between stable operations on N^* and stable N^* characteristic classes. Let θ^i be the operation corresponding to the characteristic class σ_i.

Our main result is the following relation between $\theta = \Sigma_i \theta^i$ and tom Dieck's internal Steenrod operation R [7, p. 394]. Let P^{-i} be the cobordism operation of degree $-i$ which sends $Z \longrightarrow X$ to the composition $(\mathbf{R}P^i \times Z) \longrightarrow Z \longrightarrow X$, where $\mathbf{R}P^i$ is real projective i-space.

THEOREM θ. $= PR$.

In other words, if $\alpha \in N^q(X)$, $\theta^i(\alpha) = \Sigma_j P^{i-j} R^j(\alpha)$. Since $P^{i-j} = 0$ for $j < i$ and $R^j(\alpha) = 0$ for $j > q$, this sum is finite.

It follows that θ corresponds to the "expanded square" operation in unoriented piecewise-linear cobordism [1].

This theorem is a consequence of the observation that $\bar{\sigma}_i(\xi^n) = \pi_*(e^{n+i-1})$ for $i > -n$, where π: $P(E) \longrightarrow X$ is the projection and e is the cobordism Euler class of the (dual) canonical line bundle on $P(E)$. (For $i \leqslant -n$, $\bar{\sigma}_i(\xi)$ is represented by $P(\xi \oplus \epsilon^k)$, $k = -n - i + 1$.)

REMARK. Conner and Floyd's cobordism Stiefel-Whitney classes $w_i(\xi)$ (cf. [3]) are defined by the relation $\Sigma_i(\pi^* w_i)e^{n-i} = 0$. Thus $\Sigma_i w_i \bar{\sigma}_{k-i} = 0$ for $k > 0$.

3. **Bordism operations (cf. [5]).** There are dual actions of both θ and R on smooth unoriented bordism theory N_*. If M is a closed n-manifold, and $[M] \in N_n(M)$ is the class of the identity map, $\theta^i[M]$ is represented by $\tilde{\Sigma}(df)$, where f: $M^n \longrightarrow R^{n+i-1}$ is a generic smooth map. The following result is analogous to Thom's nonembedding theorem using ordinary Steenrod operations.

COROLLARY 1. *If the locally triangulable space X immerses topologically in $\mathbf{R}^n$, then R^i is zero on $N_j(X)$ for $i + j > n$.*

The action of R^i on the bordism of a point is given by the "quadratic construction"

$$Q_k(M) = M \times M \times S^{k-1}/(x, y, s) \sim (y, x, -s), \quad k = -n - i + 1.$$

COROLLARY 2. *If M is a closed manifold, $Q_k(M)$ is cobordant with $P(TM \oplus \epsilon^k)$, $k \geqslant 1$.*

In fact, $M \times M \times D^k/(x, y, s) \sim (y, x, -s)$ minus an open tubular neighborhood of $\{[x, x, 0]\}$ is a cobordism between them. This generalizes an argument of Conner and Floyd for $k = 1$ [2, p. 62].

REMARK. "Steenrod" operations in complex cobordism can be defined in the same way as θ^i, by using complex vector bundles. Furthermore, replacing *lines* in ξ by *k-planes* in ξ yields a family of geometric operations $\theta^i_{(k)}$ for each k.

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