Abhyankar's treatment of differentials and hence spend no little space on these objects.

(3) The key points of Abhyankar's and Murthy's proofs—the projection theorems and basis theorem—should be heavily emphasized *via* examples, geometric language, pictures to indicate what is going on, and some intuition as to where the material is headed and why. For example, it would be nice to have the "cone", "plane", and "quadric" lemmas in geometric language and to have pictures and examples for all of these results. In particular, why should the reader have to wait until p. 243 for the intuition behind the word "π-quasihyperplane" when the concept itself is introduced on p. 151?

It is a shame that the authors wrote the book in such an opaque and cumbersome style. It could have been an important contribution to the literature by showing how one can apply detailed concrete computations and ideas to algebraic geometry.

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The functions of mathematical physics, by Harry Hochstadt. Pure and Applied Mathematics, Vol. 23, Wiley-Interscience, New York, 1971, xi+322 pp., $17.50.

At first sight, the theory of the special functions of mathematical physics seems to be little more than a disorganised collection of formulas. There appear to be more than fifty special functions and there is more than one definition of each one of them; for each there is a bewildering variety of differential equations, integral representations, recurrence formulas, series expansions and so on. Perhaps it is the richness of the material, or the challenge of establishing some kind of order in the chaos of seemingly unrelated results that has attracted some of the most outstanding mathematicians of the last two hundred years to work in the subject. Gauss, Euler, Fourier, Legendre, Bessel and Riemann are among the illustrious names which feature in the literature of the subject. As Wigner has written in the General Introduction to [**18**], "All of us have admired, at one time or another, the theory of the higher transcendental functions, also called special functions of mathematical physics. The variety of the properties of these functions, which can be expressed in terms of differential equations which they satisfy, in terms of addition theorems or definite integrals over the products of these functions, is truly surprising. It is surpassed only by the variety of the properties of the elementary transcendentals, that is the exponential function, and functions derived therefrom, such as the trigonometric functions. At the same time, special functions, as their full name already indicates, appear again and again as solutions of problems in theoretical physics".

It is for this latter reason that many introductory textbooks such as [**4**], [**6**], [**9**], [**13**], [**15**] and [**16**] make use of relatively unsophisticated mathematical techniques. Since they are designed for the use of physicists, engineers and other *users* of mathematics, their aim is to introduce readers to special functions by using the methods of elementary calculus and to provide them with the kind of background which will permit them to use intelligibly numerous compilations of the properties of special functions of which the most distinguished is [**2**].

Because the special functions of mathematical physics are analytic functions of their arguments, their properties are conventionally derived by means of the methods of the theory of analytic functions. The truth of this assertion is evidenced by the fact that the second half of the classic treatise on "modern" analysis [**23**] is devoted to the derivation of detailed properties of special functions by the use of function-theoretic techniques developed in the first half. It will be recalled, too, that, in addition to providing a compilation of a collection of results which would be of value to physicists who encountered Bessel functions in the course of their researches, Watson's aim in writing his monumental treatise on Bessel functions [**22**] was to develop "applications of the fundamental processes of the theory of functions of complex variables. For this purpose Bessel functions are admirably adapted; while they offer at the same time a rather wider scope for the application of the parts of the theory of functions of a real variable than is provided by trigonometrical functions in the theory of Fourier series".

In a similar way, still the best account of Legendre functions of the second kind is given in MacRobert's textbook on functions of a complex variable [**8**], and Hobson, in his treatise [**5**] on spherical and ellipsoidal harmonics, adopts Watson's philosophy.

The unifying principle came, in the end, not from analysis but from algebra—through the theory of group representations. The connexion between special functions and group representations was first pointed out by Eli Cartan [**1**], and the application of the theory of group representations to quantum mechanics in the years immediately succeeding the publication of Cartan's paper played an important role in the investigation of this connexion. For simple Lie groups (for instance $SO(3)$, the group of rotations in $\boldsymbol{R}^3$, and M(2), the Euclidean group of the plane) we can choose a basis in the representation space in such a way that the elements of some subgroup H are given by diagonal matrices with exponential functions in the principal diagonal. The remaining elements of the group can be represented in the form $h_1 g h_2$ where $h_1, h_2 \in H$ and $g(t)$ runs through a certain one-parameter manifold. It is found that the functions g coincide with the special functions of mathematical physics. For example, repesentations of M(2) are connected with the Bessel functions of the first kind, and, for a suitable choice of basis in the representation space, the matrix elements of representations of SU(2), the group of unitary unimodular matrices of the second order—closely related to $SO(3)$—are expressed in terms of Jacobi polynomials. The arguments of the special functions correspond to suitably chosen parameters of the relevant group, so that the addition theorems for the special functions merely express the multiplication laws of the group elements. Limiting cases of the addition theorems yield the differential equations which the special functions obey. The integral properties are a consequence of Frobenius' orthogonality relations for the matrix elements of irreducible representations as generalized for Lie groups by means of Hurwitz's invariant integral, as are the completeness relations. Since some of the Lie groups can be thought of as limiting cases of others, this gives rise to relations connecting the special functions representing them. For instance, the fact that M(2) can be considered as a limit of $SO(3)$ means that there is a relationship between Bessel functions and Jacobi functions.

In studying more complicated groups, such as the Lorentz group and $SO(n)$, we find that not all matrix elements of representations of these groups can be expressed in terms of "classical" special functions. It is possible to obtain such expressions only for some of the matrix elements; for the rest it is necessary to introduce functions not already met with in mathematical analysis. That these new functions possess properties as varied and as rich as those of the classical special functions has been amply demonstrated by Vilenkin [**20**].

This method of developing the theory of special functions is undoubtedly the one which should be followed by serious students of mathematics for it gives meaning to what otherwise seems chaotic. Since representations of simple groups play such a vital part in modern physics, a student of physics might well profit also from looking at special functions in this way. Fortunately there are now three good textbooks for the reader to choose from for this is the path of development chosen by Vilenkin in his encyclopaedic work [**21**] and by Willard Miller, Jr. in his more modest treatise [**11**]; but a more attractive entry to this complex of ideas—especially for the theoretical physicist—is provided by Talman in his book [**18**] based on lectures by Wigner. We should give Wigner the last word: "Naturally, the common point of view from which the special functions are here considered, and also the natural classification of their properties, destroys some of the mystique which has surrounded and still surrounds, these functions. Whether this is a loss or a gain remains for the reader to decide".

The development of the theory of Fourier series and more recently of functional analysis led to sequences of special functions being studied from the viewpoint of orthonormal bases in certain Hilbert spaces. This is the starting point of Szegö's classic work on orthogonal polynomials [**17**] and of several more recent monographs of which mention might be made of [**3**] and [**14**].

An interesting attempt to establish a different form of unified theory of special functions due to Truesdell [**19**] has been unfairly neglected, though there has been an indication recently [**10**] that it has not been forgotten entirely. Truesdell's aim is "to provide a general theory which motivates, discovers and coordinates such seemingly unconnected relations among familiar special functions" as are known to exist. Truesdell's method centers around the functional equation

$$\frac{\partial}{\partial z}F(z, \alpha) = F(z, \alpha + 1)$$

to each of whose analytic solutions there corresponds the generating function of a set of special functions. He has shown how the commonly used formulas of the theory of special functions can be derived as special cases of a handful of results derived from the study of this generating function.

Yet another approach is to regard Gauss' hypergeometric function ${}_2F_1$ and Kummer's confluent hypergeometric function ${}_1F_1$ as forming the core of the theory of special functions, since the Legendre functions and most of the classical orthogonal polynomials can be considered as ${}_2F_1$ functions, while the Bessel functions, the parabolic cylinder functions and the Coulomb wave-functions are special cases of the ${}_1F_1$ function. Natural generalizations of these functions are the generalized hypergeometric function ${}_pF_q$, MacRobert's E-function and Meijer's G-function and it is possible to set up a general theory of such functions and then to deduce the properties of the classical special functions from it. This approach, which seems singularly lacking in interest, is the one favoured by Luke [**7**]; it is not one which can be recommended to students, but then it was Luke's intention to provide a work of reference, not a textbook.

In the Preface of the book under review the author states: "While little claim can be made to originality, it is hoped that there is enough distinction in the selection of material and type of proof to throw new light on this classical subject. The aim was to present a range of topics such that both mathematicians and applied scientists with a variety of interests will find material that is useful, and mathematically and aesthetically pleasing". Whether a pure mathematician—even a classical analyst—would find the book aesthetically pleasing is a matter of doubt but there can be no doubt that an applied mathematician will find much material that is useful, not in the form of an extensive catalogue of useful results, but in the form of a careful discussion of some of the basic techniques of the subject. It does, of course, contain the main formulas likely to be used by the nonspecialist but it is concerned with techniques rather than formulas. It is well written in the sense that the author is more concerned with conveying the essential ideas of a proof than with supplying the full rigour. This is most obvious, and arguably most desirable, in his discussion of the asymptotic behaviour of the gamma function and of the Legendre functions,[1] and when he derives results for the confluent hypergeometric function by considering limiting cases of results previously established for Gauss' hypergeometric function.

The book begins with two chapters on orthogonal polynomials. The first considers general orthogonal polynomials and their role in approximation theory, ending with an account of the use of orthogonal polynomials over a curve enclosing a finite region in the complex plane to the problem of finding a holomorphic function which maps the region bijectively onto the unit disk. The classical orthogonal polynomials are introduced in the second chapter by means of the generalized Rodrigues formula and the usual material on differential equations, generating functions, etc. is presented; there are some applications too, as well as the traditional account of how Hermite polynomials are involved in the solution of Schrodinger's equation for the linear harmonic oscillator; there is an interesting and unusual discussion of the connexion between the equilibrium of a certain set of point charges and the zeros of Jacobi polynomials.

Chapter 3 is devoted to the gamma function. The three definitions—as an integral (Euler), a limit (Gauss) and an infinite product (Weierstrass) are given and shown to be equivalent. The main properties of the gamma function are then derived by means of whichever definition gives any particular result most elegantly. The chapter ends with an account of some of the more elementary properties of Mellin transforms and their applications.

In Chapter 4 there is a detailed description of the technique of determining integral expressions for the solutions of linear differential equations of the second order. The method is then illustrated with reference to the hypergeometric equation, and the principal properties of hypergeometric functions derived. Unusual in a book at this level there is a detailed account of the connexion between hypergeometric functions and the conformal mapping of curvilinear triangles, including a group theoretic discussion of the case where the sum of the angles of the triangle exceeds π.

The next two chapters deal, respectively, with Legendre functions and with spherical harmonics in higher dimensions, introduced by way of Laplace's equation. The idea of a Green's function is also developed in these chapters.

Chapter 7 is concerned with confluent hypergeometric functions and

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[1] It should be noted that an excellent treatment of asymptotic expansions of special functions is given in Olver's book [**12**] which appeared after the book under review.

Chapter 8, the longest in the book, with Bessel functions. It has in it all that could reasonably be expected in 80 pp. and certainly is an adequate preparation for the reading of Watson's treatise [**22**]. In addition to presenting standard results it has some interesting applications including one to physical optics—a subject long neglected by teachers of mathematical physics.

The final chapter deals with Hill's equation.

Each chapter has its own set of exercises and there is a bibliography of books on special functions and related topics.

In some ways the present book can be regarded as a considerably extended version of the author's earlier book [**4**] but it is difficult to avoid the impression that in this latest work he has a less clear view of the audience to which his material is directed. At one point he seems to be addressing the "applied scientists" while at other points he makes reference to concepts such as "Riemann surface" and the "Schwarz reflection principle" which demand of the reader a much higher degree of mathematical maturity than is traditionally associated with an "applied scientist"—at least one at the beginning of his career. Graduate courses in special functions are out of fashion nowadays but many practising mathematicians need to have a working knowledge of at least the elements of the subject. There is much in this book that would provide useful material for a reading course in special functions or as supplements to courses in function theory and in differential equations. It should certainly occupy a place on the shelves of applied mathematicians as a useful and succinct reference book.

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Essentials of Padé approximants, by George A. Baker, Jr., Academic Press, New York, 1975, xi+306 pp., $26.00.

The area of rational approximation and interpolation of functions has been studied intensively since the advent of electronic computers. This has brought the Padé table to the foreground and the text under review is the first pulling together of a lot of information about these tables that has appeared in the last 20 years. The texts by Perron and Wall on continued fractions, each of which devotes a chapter to the Padé table, have been among the chief references so far. A rational function $r_{m,n}(z)=p_{m,n}(z)/q_{m,n}(z)$ is of type (m, n) if $p_{m,n}(z)$ is a polynomial of degree $\leqq m$ and $q_{m,n}(z)$ a polynomial of degree $\leqq n$. $r_{m,n}(z)$ interpolates a given function $f(z)$ at the distinct points $z_1, \cdots, z_k$ if $r_{m,n}(z_i)=f(z_i)$, $i=1, \cdots, k$. If some of the points z_i coincide, say $z_1=z_2=z_3$, then it is natural to require $r_{m,n}(z_1)=f(z_1)$, $r'_{m,n}(z_1)=f'(z_1)$, and $r''_{m,n}(z_1)=f''(z_1)$ instead of $r_{m,n}(z_i)=f(z_i)$ for $i=1, 2, 3$. The case $z_1=z_2=\cdots=z_k$, i.e. $r^{(i)}_{m,n}(z_1)=f^{(i)}(z_1)$, for $i=0, 1, \ldots, k-1$ requires that $r_{m,n}(z)$ has a high order of contact with $f(z)$ at z_1. There are two classical and equivalent definitions of the (m, n) Padé approximant $R_{m,n}$ to $f(z)$ at $z=0$:

1. find the unique rational function $R_{m,n}$ in lowest terms such that $f(z)-R_{m,n}(z)=O(z^k)$, $k=$maximum, and
2. find polynomials $P_{m,n}$ and $Q_{m,n}$ such that $Q_{m,n}(z)f(z)-P_{m,n}(z)=O(z^{m+n+1})$, and let $R_{m,n}$ be $P_{m,n}/Q_{m,n}$ in lowest terms.

In definition 1, $R_{m,n}$ depends on $m+n+1$ parameters and one would