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EXTENSIONS OF C^*-ALGEBRAS AND ESSENTIALLY n-NORMAL OPERATORS

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Let H be a separable, infinite dimensional, complex Hilbert space, and let $L(H)$ be the algebra of all (bounded, linear) operators on H. The ideal of all compact operators on H will be denoted $K(H)$, and the (Calkin) quotient algebra $L(H)/K(H)$ will be denoted by $Q(H)$. Given a C^*-algebra A with identity, an extension τ of $K(H)$ by A (or simply an extension τ by A) is, by definition, an identity preserving injective *-homomorphism $\tau: A \longrightarrow Q(H)$. In [2] a complete classification of all extensions of $K(H)$ by any abelian separable C^*-algebra (with the natural equivalence relation) was obtained. As indicated in [2] and [3], if one wishes to attack the classification problem for extensions by noncommutative C^*-algebras, it is reasonable to restrict attention to separable ones. Henceforth, A will be assumed to be a separable C^*-algebra with identity. Also, we shall denote by π the canonical quotient map from $L(H)$ onto $Q(H)$. An extension τ by A will be said to be trivial if there exists a faithful nondegenerate *-representation $\sigma: A \longrightarrow L(H)$ such that $\tau = \pi\sigma$. It readily follows that trivial extensions by A always exist. We shall say that two extensions τ_1 and τ_2 by A are *equivalent* and we write $\tau_1 \approx \tau_2$ if there exists an operator W in $L(H)$ such that πW is a unitary element of $Q(H)$ and $\tau_1 A \pi W = \pi W \tau_2 A$, for every A in A. (In the terminology of [2] τ_1 and τ_2 are called weakly equivalent.) The set of all equivalence classes of extensions by A, under this equivalence relation, will be denoted by Ext A. Following the pattern of [2] and [3] we define a binary operation on Ext A as follows: let τ_1 and τ_2 be two extensions by A and let $\tau': A \longrightarrow Q(H) \oplus Q(H)$ given by $\tau' = \tau_1 \oplus \tau_2$; after identifying $Q(H) \oplus Q(H)$ with a C^*-subalgebra of $Q(H)$, we then obtain an extension τ by A whose equivalence class $[\tau]$ will be called the sum of $[\tau_1]$ and $[\tau_2]$.

The following theorem generalizes [2, Theorem 9.2].

THEOREM 1. *If every irreducible *-representation of A is finite dimensional, then* Ext A *is an abelian semigroup whose identity is the equivalence class of all trivial extensions by A. Moreover, if A also satisfies the property that for every identity preserving completely positive map $\varphi: A \longrightarrow Q(H)$, there exists an identity preserving completely positive map $\psi: A \longrightarrow L(H)$ such that $\varphi = \pi\psi$, then* Ext A *is a group.*

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THEOREM 2. *Suppose that every irreducible *-representation of* $\mathcal{A}$ *is at most n-dimensional, for some positive integer n. If* τ_1 *and* τ_2 *are two equivalent extensions by* $\mathcal{A}$, *then there exists an operator W in* $\mathcal{L}(\mathcal{H})$ *such that* πW *is unitary, the absolute value of the index of W is less than n, and* $\tau_1 A \pi W = \pi W \tau_2 A$ *for every A in* $\mathcal{A}$.

Notice that $\mathcal{A}$ satisfies the hypothesis of Theorem 2 if and only if $\mathcal{A}$ is n-normal; that is $\Sigma_\sigma \epsilon_\sigma A_{\sigma(1)} \cdots A_{\sigma(2n)} = 0$, where σ ranges over the permutation group of $2n$ elements, $A_{\sigma(j)} \in \mathcal{A}$, $1 \leqslant j \leqslant 2n$, and ϵ_σ is the sign of the permutation σ.

An operator T in $\mathcal{L}(\mathcal{H})$ will be called algebraically n-normal [essentially algebraically n-normal] if the C^*-algebra $C^*(T)$ generated by T and $1_{\mathcal{H}}$ [$\pi C^*(T)$] is n-normal, that is, any of its irreducible *-representations is at most n-dimensional. Furthermore, an operator T in $\mathcal{L}(\mathcal{H})$ will be called n-normal [essentially n-normal] if there exists a unitary transformation $U: \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathbf{C}^n$ such that UTU^* [$\pi_{\mathcal{H}\otimes\mathbf{C}^n}(UTU^*)$] is a matrix whose entries are commuting normal elements of $\mathcal{L}(\mathcal{H})$ [$\mathcal{Q}(\mathcal{H})$]. It is well known that T is algebraically n-normal if and only if $T = \Sigma_{k=1}^n \oplus T_k$, where T_k is k-normal, $1 \leqslant k \leqslant n$. (In this direct sum some direct summands might be either absent or finite dimensional.) At the present time we do not know whether the corresponding version of this result, in the Calkin algebra setting, for essentially algebraically n-normal operators holds. (Note that the "if" part is true.)

Let $\mathcal{M}_n$ be the algebra of all complex $n \times n$ matrices, which we identify with the algebra of all linear operators on the finite dimensional complex Hilbert space $\mathbf{C}^n$ via the standard basis of $\mathbf{C}^n$. Following [5] and [6], given an operator T in $\mathcal{L}(\mathcal{H})$ we define the reducing essential $n \times n$ matricial spectrum of T by (also cf. [4])

$$R_e^n(T) = \{L \in \mathcal{M}_n\colon \text{ there exists a nondegenerate } n\text{-dimensional *-representation } \varphi\colon C^*(T) \longrightarrow \mathcal{M}_n \text{ such that } \mathcal{K}(\mathcal{H}) \cap C^*(T) \subset \ker \varphi \text{ and } \varphi(T) = L\}.$$

We note that if $X \subset \mathcal{M}_n$ and there exists T in $\mathcal{L}(\mathcal{H})$ such that $R_e^n(T) = X$, then X satisfies the following property:

(*) X is compact, it is the union of unitary equivalence classes in $\mathcal{M}_n$, and if $L_j \oplus M_j \in X$ where $L_j \in \mathcal{M}_{k_j}$, $1 \leqslant j \leqslant m$, and $\Sigma_{j=1}^m k_j = n$, then $\Sigma_{j=1}^m \oplus L_j \in X$, $m = 1, 2, \cdots$.

Furthermore, if T is essentially n-normal, then $X = R_e^n(T) \neq \varnothing$.

A subset X of $\mathcal{M}_n$ will be called hypoconvex if $X \neq \varnothing$ and X satisfies (*).

THEOREM 3. *Let X be a hypoconvex set in* $\mathcal{M}_n$. *Then there exists an n-normal operator T in* $\mathcal{L}(\mathcal{H})$ *such that* $R_e^n(T) = X$.

Given a hypoconvex set X in M_n, let $EN_n(X)$ denote the set of equivalence classes of all essentially n-normal operators T in $L(H)$ such that $R_e^n(T) = X$, with the following equivalence relation: $T_1 \sim T_2$ if and only if there exist K in $K(H)$ and an isometry V in $L(H)$ with deficiency less than n such that $V^*T_1V = T_2 + K$. (That this is an equivalence relation can be proved with the aid of [6, Theorem 4.4].) Now we define a binary operation on $EN_n(X)$ suggested by the one defined previously on Ext A. Given T_1 and T_2 two essentially n-normal operators with $[T_1]$, $[T_2] \in EN_n(X)$, we let $[T_1] + [T_2] = [T_1 \oplus T_2]$, where $T_1 \oplus T_2$ is identified with an essentially n-normal operator in $L(H)$.

The following theorem generalizes a result of [1].

THEOREM 4. *$EN_n(X)$ is an abelian group whose identity consists of the class of all operators in $L(H)$ whose compression to some subspace of codimension less than n is a compact perturbation of an n-normal operator.*

Let X be a hypoconvex set in M_n and let $\chi: X \longrightarrow X$ be the identity map on X. Further, let $C_\chi^*(X)$ be the C^*-subalgebra of $C(X) \otimes M_n$, generated by χ and 1. Let $\text{Ext}_\chi^n(X)$ be the subset of Ext $C_\chi^*(X)$ consisting of those equivalence classes of extensions τ such that $\tau\chi$ is an n-normal element in $Q(H)$ (that is, there exists an essentially n-normal operator T in $L(H)$ such that $\pi T = \tau\chi$).

THEOREM 5. *$\text{Ext}_\chi^n(X)$ is an abelian group which is isomorphic to $EN_n(X)$.*

Given a hypoconvex set X in M_n let $F(X)$ denote the set of all invertible continuous functions $f: X \longrightarrow M_n$ such that $f \in C_\chi^*(X)$. Let $\Pi_\chi^n(X)$ be the set of all homotopy equivalence classes of $F(X)$. Then $\Pi_\chi^n(X)$ has a natural group structure.

THEOREM 6. *There exists a natural homomorphism $\iota_X: EN_n(X) \longrightarrow \text{Hom}(\Pi_\chi^n(X), \mathbf{Z})$, where $\mathbf{Z}$ is the set of integer numbers.*

As R. G. Douglas pointed out to us the homomorphism ι_X will not be injective in general because $\text{Hom}(\Pi_\chi^n(X), \mathbf{Z})$ is torsion free while the group $EN_n(X)$ may have elements of finite order. Thus, it is natural to ask: What is the obstruction to the injectivity of ι_X? Notice that if ι_X is injective for a given hypoconvex set $X \subset M_n$, then from Theorem 6 it follows that $\text{Hom}(\Pi_\chi^n(X), \mathbf{Z})$ is a complete set of invariants up to the equivalence relation $\sim$ for the class of essentially n-normal operators T such that $R_e^n(T) = X$.

REFERENCES

1. W. Arveson, *A note on essentially normal operators*, Proc. Roy. Irish Acad. Sect. A 74 (1974), 143–146.

2. L. Brown, R. Douglas and P. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 58–128.

3. ———, *Extensions of C^*-algebras, operators with compact self-commutators, and K-homology*, Bull. Amer. Math. Soc. 79 (1973), 973–978. MR 49 #11265.

4. John W. Bunce and Norberto Salinas, *Completely positive maps on C^*-algebras*

and the left matricial spectra of an operator (preprint)

5. Carl Pearcy and Norberto Salinas, *Finite dimensional representations of C^*-algebras and the reducing matricial spectra of an operator,* Rev. Roumain. Math Pures Appl. **20** (1975), 567–598.

6. ———, *The reducing essential matricial spectra of an operator,* Duke Math. J. **42** (1975), 423–434.

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