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FLAT HOMOLOGY

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In this note we define "homology groups" relative to the flat site, and list some of their properties, in the case that the base scheme is algebraic over a field.

X_{fl} denotes the big f.p.p.f. site over a scheme X and $S(X_{fl})$ the corresponding category of sheaves. $S = \operatorname{spec} k$, where k is a field of characteristic p. $\mathsf{A}(al)$ denotes the category of commutative algebraic group schemes over S and $\mathsf{A}(u, f) \supset \mathsf{A}(u) \supset \mathsf{A}(uf) \supset \mathsf{A}(f)$ the subcategories consisting of those affine groups which are respectively unipotent or finite, unipotent, unipotent and finite, finite. The letter A always stands for one of these categories and Pro-A for the corresponding pro-category. The notations for derived categories are as in [6].

1. THEOREM (Universal Coefficient Theorem). *For any morphism* π: $X \to S$ *of finite type and any* A, *there exists a complex* $L_{\bullet}(X/S, \mathsf{A})$ *in* $K^-(\text{Pro-}\mathsf{A})$ *such that*:

(a) $L_s(X/S, \mathsf{A})$ *is a projective object, all* s;

(b) $\operatorname{Hom}_{\text{Pro-}\mathsf{A}}(L_{\bullet}(X/S, \mathsf{A}), N) \xrightarrow{\approx} \mathbf{R}\pi_* N_X$ *in* $D^+(S(S_{fl}))$ *for all* N *in* A.
Moreover, $L_{\bullet}(X/S, \mathsf{A})$ *is unique, up to isomorphism, in* $K^-(\text{Pro-}\mathsf{A})$.

PROOF. Choose a conservative family of points for X_{fl}, and let $C^{\bullet}(F)$ be the corresponding Godement resolution of a sheaf F [1, XVII 4.2]. Choose L_s to pro-represent the functor $N \mapsto \Gamma(X, C^s(N_X))$: $\mathsf{A} \longrightarrow \mathsf{A}b$.

2. COROLLARY. *Write* $H_s(X/S, \mathsf{A})$ *for* $H_s(L_{\bullet}(X/S, \mathsf{A}))$. *There is a spectral sequence*

$$\operatorname{Ext}^r_{\text{Pro-}\mathsf{A}}(H_s(X/S, \mathsf{A}), N) \Rightarrow H^{r+s}(X_{fl}, N_X) \quad \textit{for all } N \textit{ in } \mathsf{A}.$$

3. DEFINITION. $L_{\bullet}(X/S, \mathsf{A})$ is the flat homology complex of X/S relative to A, and $H_s(X/S, \mathsf{A})$ is the sth flat homology group.

4. REMARKS. (a) Theorem 1 is basically as conjectured by Grothendieck [5, p. 316].

(b) $L_{\bullet}(X/S, \mathsf{A})$ and $H_s(X/S, \mathsf{A})$ are covariant functors in X/S.

(c) *If* ω_0: $\mathsf{A}(al) \longrightarrow \mathsf{A}(f)$ is the functor taking a group scheme to its maximal finite quotient, then $\omega_0(L_{\bullet}(X/S, \mathsf{A}(al))) = L_{\bullet}(X/S, \mathsf{A}(f))$. Thus there

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is a third-quadrant spectral sequence $\omega_r(H_s(X/S, \mathsf{A}(al))) \Rightarrow H_{r+s}(X/S, \mathsf{A}(f))$ where $\omega_r = L^r\omega_0$.

5. THEOREM. *Assume k to be algebraically closed and let M be the functor taking a group scheme to its Dieudonné module (in the sense of* [4, III]). *Then*

$$M(L_\bullet(X/S, \mathsf{A}(u, f))) = \mathbf{H}\ (X_{Zar}, \underset{\rightarrow}{W}) \oplus (\mathbf{H}^\cdot(X_{fl}, \mu_{p^\infty}) \otimes_{\mathbf{Z}} W(k)),$$

where W_n is the group scheme of Witt vectors of length n and $\underset{\rightarrow}{W} = \varinjlim W_n(O_X)$ and $W(k) = \varprojlim W_n(k)$.

PROOF. Immediate from the definitions of $L_\bullet$ and M.

6. COROLLARY. $M(H_s(X/S, \mathsf{A}(u, f))) = H^s(X_{Zar}, \underset{\rightarrow}{W}) \oplus H^s(X_{fl}, \mu_{p^\infty}) \otimes_{\mathbf{Z}} W(k)$.

PROOF. "$\varinjlim$" W_n and "$\varinjlim$" μ_{p^n} behave as injectives in A.

7. REMARK. $M(H_1)$ is equal to the group $I(X)$ studied in [7, §4].

8. THEOREM. *Assume k to be algebraically closed and X/S to be proper. Then $L_\bullet(X/S, \mathsf{A}(u))$ is isomorphic (in K^-(Pro-$\mathsf{A}(u)$)) to $L_\bullet(X/S, \mathsf{A}(uf))$.*

PROOF. $H_s(X/S, \mathsf{A}(u)) \in$ Pro-$\mathsf{A}(uf)$ for otherwise $H^s(X, O_X)$ would have infinite dimension over k.

9. THEOREM. *Write $N^\vee$ for the formal group associated to an affine group scheme N by Cartier duality (see* [4, II.4]), *and write $H_s^\vee$ for $H_s(L_\bullet)^\vee = H^s(L_\bullet^\vee)$ where $L_\bullet = L_\bullet(X/S, \mathsf{A}(u))$. Then $H_s^\vee$ is a connected formal group of finite-type (see* [4, *p.* 35]) *and represents the functor of finite S-schemes.*

$$T \longmapsto \mathrm{Ker}(\Gamma(T, R^s\pi_*\mathbf{G}_m) \longrightarrow \Gamma(T_{red}, R^s\pi_*\mathbf{G}_m)).$$

PROOF. Regard $U = \mathrm{Ker}(\mathbf{G}_{m,T} \longrightarrow \mathbf{G}_{m,T_{red}})$ as a sheaf on T_{red}, and use (8).

10. COROLLARY. *Write* $\Phi^s(T) = \mathrm{Ker}(H^s(X_T, \mathbf{G}_m) \longrightarrow H^s(X_{T_{red}}, \mathbf{G}_m))$. *If Φ^{s-1} is a formally smooth functor then Φ^s is represented by a formal group.*

PROOF. Immediate from the theorem.

11. REMARKS. (a) Intuitively (9) says that $L_\bullet^\vee$ represents $\mathbf{R}^\cdot\pi_*\mathbf{G}_m$ infinitesimally.

(b) Generalizations of (10), but not (9), may be found in [2].

12. THEOREM. *Assume that k is algebraically closed, X is projective and smooth over k, and $p > \dim(X)$. Then*

$$\mathrm{Hom}_W(K/W, M(H_s(X/S, \mathsf{A}(f)))) \otimes_W K \xrightarrow{\approx} (H^s(X/W, O_{X/W}) \otimes_W K)_{[0,1]}$$

as F-isocrystals, where $W = W(k)$, K = field of fractions of W, and the right-hand term is the part of crystalline cohomology with slopes between 0 *and* 1 (*inclusive*).

PROOF. Follows from [3] and (6).

13. REMARKS. (a) The last theorem states that (modulo torsion) the knowledge of the flat cohomology of finite constant group schemes on X is equivalent to the knowledge of the part of crystalline cohomology with slopes between 0 and 1.

(b) (12) differs from the "hope" expressed by Grothendieck [5, p. 316].

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