

## A PARTIAL TOPOLOGICAL CLASSIFICATION FOR STABLE MAP GERMS

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1. **Introduction.** The two theories of  $C^0$  and  $C^\infty$ -stability for smooth functions  $f: M \rightarrow N$  between smooth manifolds [2], [3], [5] both provide for appropriate dimensions a classification for a dense subset in the space of smooth mappings. In this note we announce a result which partially describes how distinct  $C^\infty$ -stable map germs  $f: \mathbf{R}^n \rightarrow \mathbf{R}^p$  with  $n \leq p$  are related under the weaker notion of topological equivalence. We say after [1] that a stable map germ  $f$  is of *discrete algebra type* if there are only a finite number of germ types nearby with associated algebra an algebra in the same number of generators as  $Q(f)$  (i.e. have same  $\Sigma_i$  type). This is the largest class of stable map germs which do not require moduli for their classification. It includes not only the stable map germs in the nice dimensions, but also simple stable map germs (in the same sense as used by Arnold for functions).

Following a conversation with Andre Galligo, it became clear that the best way to describe this partial topological classification is to use the Hilbert-Samuel function of the associated algebra.

**THEOREM 1.** *For stable map germs of discrete algebra type  $f$ , the Hilbert-Samuel function of  $Q(f)$  is a topological invariant.*

The topological classification actually gives a stronger result for a number of cases, namely, that the complex algebra type is a topological invariant. The author hopes to complete this result in a subsequent paper.

Lastly, the author wishes to thank both Andre Galligo and John Mather.

2. **Germs of discrete algebra type.** These types were essentially determined by Mather [2-VI]. He determined where moduli first appeared in each  $\Sigma_i$  type except one. We recall that if  $f: \mathbf{R}^n \rightarrow \mathbf{R}^p$  is of type  $\Sigma_i$  at 0, then letting  $K = \ker D_0 f$ ,  $C = \text{coker } D_0 f$ , we have  $\dim K = i$ ; and there is a second intrinsic derivative  $\tilde{D}_0^2 f: S^2 K \rightarrow C$ . Then,  $f$  is of type  $\Sigma_{i,(j)}$  if  $\dim \ker(\tilde{D}_0^2 f) = j$ . If  $\ker(\tilde{D}_0^2 f) = S^2 K$ , then we can define  $\tilde{D}_0^3 f: S^3 K \rightarrow C$ . We can repeat this until  $\ker D_0^l f \neq S^l K$ . This

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gives types  $\Sigma_{i,i,\dots,i,(j)(l-1i's)}$  where  $j = \dim \ker \widetilde{D}_0^l(f)$ . Then, moduli first occur as follows: (i) for  $\Sigma_2$  types, the  $\Sigma_{2,1}$ -type has the modulus

$$\mathbf{R}[[x, y]] / (x^2 \pm y^4, xy^3 + ay^5)$$

and the  $\Sigma_{2,2,(2)}$  has moduli; (ii) for  $\Sigma_3$ -types,  $\Sigma_{3,(3)}$  has moduli and the  $i\beta$ -type of  $\Sigma_{3,(2)}$  also has the modulus

$$\mathbf{R}[[x, y, z]] / (x^2 + axy^2 + y^4, xz + y^4, yz, z^2 + xy^2);$$

(iii) lastly,  $\Sigma_{r,(2)}$  for  $r \geq 4$  has moduli.

In addition to the types determined by Mather, there is one additional  $\Sigma_{2,1}$ -type, two additional  $\Sigma_{3,(2)}$ ,  $i\epsilon 2$ -types and lastly the types  $\Sigma_{3,(2)}$ ,  $i\beta$  not classified by Mather.

**3. Principal results.** If a map germ  $f: \mathbf{R}^n \rightarrow \mathbf{R}^p$  is of type  $\Sigma_{i,(j)}$ , and  $h(m)$  denotes the Hilbert-Samuel function of  $Q(f)$ , then

$$h(m) = \dim_{\mathbf{R}} C_0^\infty(\mathbf{R}^n) / (f^* M_p \cdot C_0^\infty(\mathbf{R}^n) + M_n^{m+1}) = \dim_{\mathbf{R}} Q_m(f)$$

so that  $h(1) = i + 1$  and  $h(2) = 1 + i + j$ . Thus, to say that  $h(1)$  and  $h(2)$  are topological invariants is equivalent to saying that the  $\Sigma_{i,(j)}$ -type is a topological invariant. For this, we have

**THEOREM 2.** *For  $C^\infty$ -stable map germs  $f: \mathbf{R}^n \rightarrow \mathbf{R}^p$ , the  $\Sigma_i$ -type is a topological invariant. If, moreover,  $p \geq n + \binom{i}{2}$  then the  $\Sigma_{i,(j)}$ -type is also a topological invariant.*

It was pointed out to the author by John Mather that the first part of this theorem has been proven in a different context by Robert May [4].

**COROLLARY 1.** *If  $p < n + \binom{i}{2}$ , let  $s = \binom{i}{2} - (p - n)$ . Then, if  $\Sigma_{i,(s)}$  and  $\Sigma_{i,(s+1)}$ -types are topologically distinct, the  $\Sigma_{i,(j)}$ -type is a topological invariant.*

**COROLLARY 2.** *If  $p \geq n + k - 1$  then for stable map germs of types  $\Sigma_{2,\dots,2,(j)}$  ( $l$  factors),  $l \leq k$ , the  $\Sigma_{2,\dots,2,(j)}$  ( $l$  factors)-type is a topological invariant.*

Together with the topological invariance of the  $\Sigma_{i,(j)}$ -type we also have from [1] that  $\delta(f) = \dim_{\mathbf{R}} Q(f)$  is a topological invariant for stable map germs of discrete algebra type. Then, except for germs of types  $\Sigma_{2,(2)}$  ( $\Sigma_{2,0}$  and  $\Sigma_{2,1}$ ) and  $i\epsilon 2$  and  $i\beta$ -types in  $\Sigma_{3,(2)}$  the Hilbert-Samuel function is determined by the  $\Sigma_{i,(j)}$ -type and  $\delta$ .

To handle the cases  $\Sigma_{2,0}$ ,  $\Sigma_{2,1}$  and the  $i\epsilon 2$ ,  $i\beta$  types of  $\Sigma_{3,(2)}$  we have to study the structure of nearby singularity types.

We say that a germ-type  $g$  is nearby a stable map germ  $f$  if for any representative  $f_1$  of  $f$  and any neighborhood  $U$  of the source 0 there is  $x \in U$  such that the germ of  $f_1$  at  $x$  is equivalent to  $g$ .

As sufficiently nearby  $g$  are also stable, we can examine them by examining their algebras. Determining which algebras occur nearby can be difficult. However, we can say

**PROPOSITION 1.** *If  $f$  is a stable map germ  $f: \mathbf{R}^n \rightarrow \mathbf{R}^p$  with  $Q \simeq Q(f)$ . Then, if  $Q'$  is a quotient algebra of  $Q$  with  $p - n \geq -i(Q')$ , then there is a stable map germ-type  $g$  nearby  $f$  with  $Q(g) \simeq Q'$ .*

**PROPOSITION 2.** *If  $f$  is a stable map germ of type  $\Sigma_k$  so that  $Q_2(f) \simeq \mathbf{R}[[x_k]]/V + M^3$  where  $V \subset S^2\langle x_1, \dots, x_k \rangle$  is of dim  $v$ . Then, if  $g$  is nearby  $f$  and of type  $\Sigma_k$  then  $Q_2(g) \simeq \mathbf{R}[[x_k]]/W + M^3$  where  $W \subset S^2\langle x_1, \dots, x_k \rangle$  of dim  $\leq p - n + k$  and  $W \supset V_1$  with  $V_1$  near  $V$  in  $G_v(S^2\langle x_1, \dots, x_k \rangle)$ .*

We can also give a similar result for nearby  $\Sigma_{k-1}$ -types.

Then Propositions 1 and 2 can be used to show that I–II–III-types, the IV–V-types, and the  $\Sigma_{2,1}$ -types are distinct. For the types in I–II–III $_{a,b}$ , in  $\Sigma_{2,0}$ , the  $\Sigma_{2,1}$ -types, and the  $1\epsilon 2$  and  $ii\beta$ -types in  $\Sigma_{3,(2)}$ , we make a more detailed study of the singularity sets using the above results.

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