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OPEN THREE-DIMENSIONAL MANIFOLDS WITH FINITELY GENERATED FUNDAMENTAL GROUPS

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Recent results of G. P. Scott [5] and T. W. Tucker [6] indicate that a 3-manifold with a finitely generated fundamental group is in various senses close to being compact. The results announced in this paper are further investigations into the relations between these two properties for open 3-manifolds. With a few additional complications these results also hold for noncompact 3-manifolds with boundary.

THEOREM 1. *Suppose M is an open, connected 3-manifold, $\pi_1(M)$ is finitely generated, M contains no infinite collection of disjoint fake 3-cells, and M contains no 2-sided projective planes. Then M is homotopy equivalent to the interior of a compact 3-manifold with a tame, closed, 0-dimensional subset deleted. The compact 3-manifold is homeomorphic to a submanifold of M.*

Theorem 1 is an extension of Theorem 3.2 of [1]. In general, of course, the homotopy equivalence will not be proper: Whitehead's example [7] of a contractible, open 3-manifold shows that even in simple cases, the structure at infinity can be quite complicated.

After finding a maximal, finite collection of disjoint fake 3-cells in M and replacing them with 3-cells, the conclusion of Theorem 1 follows from the results of [4] that are stated as Theorems 2 and 3 below.

DEFINITIONS. Suppose M is an open, connected 3-manifold. A *submanifold* of M is a 3-manifold embedded as a polyhedral subset of M. A *punctured collar* on the boundary of a submanifold S of M is a submanifold C in $M - \text{Int}(S)$ such that $C \cap S = \partial S$ and $(C, \partial S)$ is homeomorphic to $(\partial S \times I - \text{Int}(B^3), \partial S \times \{0\})$ where B^3 is a union of disjoint 3-cells in $\text{Int}(\partial S \times I)$. A *nucleus* of M is a compact submanifold S such that any compact subset of M can be engulfed by adding a punctured collar to ∂S and then attaching 1-handles to the boundary of the resulting submanifold.

THEOREM 2. *Suppose M is an open, connected 3-manifold, $\pi_1(M)$ is finitely generated, and M contains no fake 3-cells and no 2-sided projective planes. Then M has a nucleus.*

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Theorem 2 is modeled after a similar result of McMillan [3]. The proof of Theorem 2 consists of first showing that there is a compact submanifold S of M satisfying the following six properties, and then showing that such a submanifold is a nucleus. The six properties in fact characterize nuclei that are either 3-cells or have incompressible boundaries.

(1) Punctured 3-cells are the only compact, connected submanifolds of $M - \mathrm{Int}(S)$ that are bounded by 2-spheres.

(2) Either S is a 3-cell or ∂S is incompressible in M.

(3) Any indecomposable noninfinite-cyclic subgroup of $\pi_1(M)$ is conjugate to a subgroup of the fundamental group of a component of S.

(4) No component of $M - \mathrm{Int}(S)$ is compact.

(5) S is not contained in any compact submanifold of M with incompressible boundary and fewer non-2-sphere boundary components.

(6) Every 2-sphere in $M - S$ separates the component of $M - \mathrm{Int}(S)$ containing it, and one of the resulting complementary domains contains no component of ∂S.

THEOREM 3. *Suppose M is an open, connected 3-manifold, $\pi_1(M)$ is finitely generated, and M contains a nucleus. Then $M \times \mathbf{R}$ is homeomorphic to $(U \cup (C - P)) \times \mathbf{R}$ where U is homeomorphic to a compact submanifold of M obtained by adding certain 1-handles to an appropriate nucleus of M, C is an open collar on ∂U, and P is a tame, closed, 0-dimensional subset of* $\mathrm{Int}(C)$. *Specifically any compact subset of P can be written as the intersection of a nested sequence of unions of disjoint 3-cells in* $\mathrm{Int}(C)$.

Theorem 3 is proved by writing M as the union of a nested sequence of submanifolds formed by adding punctured collars and 1-handles to a nucleus. The construction in Scott's paper [5] is used to differentiate between 1-handles that are essential and those that are redundant. The remainder of the proof follows in outline McMillan's technique [2] of showing that the product of an irreducible, simply connected, open 3-manifold with $\mathbf{R}$ is the interior of a 4-cell.

COROLLARY. *In the situation stated in Theorem 3, suppose also that $\pi_2(M) \cong (0)$. Then $P = \varnothing$ and $M \times \mathbf{R}$ is homeomorphic to* $\mathrm{Int}(U) \times \mathbf{R}$, *which is homeomorphic to the interior of the compact 4-manifold $U \times I$.*

COROLLARY. *Suppose P and P' are Cantor sets embedded in the 3-sphere S^3 so that their complements have finitely generated fundamental groups. Then the complements are simply connected and $(S^3 - P) \times \mathbf{R}$ is homeomorphic to* $(S^3 - P') \times \mathbf{R}$.

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