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VERSAL UNFOLDINGS OF G-INVARIANT FUNCTIONS

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1. We announce here some results on equivariant local differential analysis. The proofs will appear elsewhere [7]. We consider a compact Lie group G, acting orthogonally on R^n. $C^\infty(x)$ (respectively $C^\infty(R^n)$) will denote the ring of germs of C^∞ functions around $0 \in R^n$ (the ring of C^∞ functions of R^n). The germ of R^n at 0 will be denoted by X. $C^\infty(x)^G$, $C^\infty(R^n)^G$ will denote the G-invariant germs (functions). We shall consider parameter (germs of) spaces $U, V, \ldots,$ on which G acts, by definition, trivially.

If $f(x) \in C^\infty(x)^G$, an *unfolding* of $f(x)$ is an $F(x, u) \in C^\infty(x, u)^G$ such that $F(x, 0) \equiv f(x)$. The unfolding $F(x, u)$ is *versal*, if any other unfolding of $f(x)$, $H(x, v) \in C^\infty(x, v)^G$, can be induced from F, by a commutative diagram

$$\begin{array}{ccc} X \times V & \xrightarrow{\Phi} & X \times U \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & U \end{array}$$

such that:

(a) $\Phi, \varphi \in C^\infty$,

(b) Φ is G-equivariant,

(c) $\Phi | X \times 0 \equiv \operatorname{id} X$,

(d) $H = F \circ \Phi$.

G also acts on smooth vector-fields on $X(R^n)$. We consider the *invariant* (germs of) vector-fields $\Gamma^\infty(TX)^G \subset \Gamma^\infty(TX)$ i.e., fields such that $g\xi(x) = Tg(\xi(x)) = \xi(gx)$. $\Gamma^\infty(TX)^G$ is a $C^\infty(x)^G$-module moreover, if $f(x) \in C^\infty(x)^G$, the subset

$$J_G(f) = \{df(\xi), \xi \in \Gamma^\infty(TX)^G\} \subset C^\infty(x)^G.$$

is an ideal, called the *G-jacobian ideal of f.* We shall assume that f is given, and that $\dim_R C^\infty(x)^G/J_G(f) < \infty$.

By definition $F(x, u) \in C^\infty(x, u)^G$, unfolding of f, is *infinitesimally versal* if the images of $\partial F(x, 0)/\partial u_1, \ldots, \partial F(x, 0)/\partial u_k$ in $C^\infty(x)^G/J_G(f)$ generate the R-vector space $C^\infty(x)^G/J_G(f)$.

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THEOREM 1. *If the unfolding* $F(x, u) \in C^\infty(x, u)^G$ *(of* $f(x) \in C^\infty(x)^G$*) is infinitesimally versal, it is versal.* □

This is a generalization of a result of J. Mather [5], R. Thom [16], V. M. Zakalyukin [14], F. Sergeraert [10], G. Lassalle [3], and others.

This theorem should be useful for "catastrophy theory in the presence of symmetry" [11], [12].

2. The main ingredient for proving Theorem 1 is the *equivariant preparation theorem,* which we describe now.

Suppose G (compact Lie group) acts orthogonally on R^n, R^p; the germs of these two spaces, around 0, will be denoted by X, Y.

We consider a germ of smooth map $f \in C^\infty(X, Y)$ which is equivariant: $f(gx) = gf(x)$. Then f induces a local ring homomorphism $C^\infty(x)^G \xleftarrow{f^*} C^\infty(y)^G$.

THEOREM 2. *If M is a finitely generated $C^\infty(x)^G$-module, such that* $\dim_R M/f^*\mathfrak{M}C^\infty(y)^G \cdot M < \infty$, *then M is also finitely generated as a $C^\infty(y)^G$-module.* □

This is a generalization of a theorem of B. Malgrange [4] and J. Mather [6].

3. This paragraph provides some examples for Theorem 1.

With G compact as before we consider the algebra of G-invariant polynomials $R[x]^G$. By a classical result of Hilbert [2], [13], this algebra is finitely generated, i.e. there is a *polynomial* map $y = \rho(x)$ ($R^n \xrightarrow{\rho} R^p$) (given by finitely many homogenous polynomials, of positive degree), such that $R[x]^G \xleftarrow{\rho^*} R[y]$ is *surjective.* It had been conjectured, for some time, that this is still true in the C^∞ case. In fact G. Glaeser [15] had proved it for $G =$ the symmetric group, and for some time at least the local case for finite G has been known to result from the preparation theorem (see for example [1]). Note also that there is a way to work along the diagonals and go from the local to the global case. Now, the general compact case has been proved by G. Schwarz [9], and it is this result which makes the present paper possible. We hope to be able to complete the details of a different proof, in some future (including, possibly, the C^k-case). Since Hilbert's XIVth problem is solved negatively, the noncompact case is hopeless.

Now if ξ is a smooth G-invariant vector field on R^n, one has in a natural way, a *direct image* of ξ: $\rho_*\xi$, which is a continuous vector field on the semialgebraic subset $\rho R^n \subset R^p$.

PROPOSITION 3. *If* $\xi \in \Gamma^\infty(TR^n)^G$, *then there is a smooth* (C^∞) *vector field* $\eta \in \Gamma^\infty(TR^p)$ *such that* $\eta | \rho R^n \equiv \rho_*\xi$. □

The same result is true for germs, and we deduce that if $\varphi(y) \in C^\infty(y)$, and $J(\varphi) \subset C^\infty(y)$ is the usual jacobian ideal of φ, then $\rho^*J(\varphi) \supset J_G(\rho^*\varphi)$. (Note that $\rho^*\varphi \in C^\infty(x)^G$.) This leads to one way of finding elements of finite codimension in $C^\infty(x)^G$. A better way is given by the following

PROPOSITION 4. *Let $f(x) \in C^\infty(x)^G \subset C^\infty(x)$ such that*

$$\dim_R C^\infty(x)/J(f) < \infty.$$

Let $\varphi_1(x), \ldots, \varphi_k(x) \in C^\infty(x)$ be generators of $C^\infty(x)/J(f)$, as a vector space. Then $C^\infty(x)^G/J_G(f)$ is a finite dimensional vector space, generated by the averages of the φ_i's:

$$\psi_i(x) = \int_G \varphi_i(gx)\, d\mu(g) \in C^\infty(x)^G. \quad \square$$

Here $d\mu(g)$ is the Haar measure of G. The general idea behind all this is that once one has a smooth version of Hilbert's finiteness theorem from the classical invariant theory, the Thom-Mather type theory of singularities can be extended to the case when a *compact* Lie group is operating. We plan to develop stability theory on these lines (see also [8]).

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