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A SUFFICIENT CONDITION FOR k-PATH HAMILTONIAN DIGRAPHS

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A directed graph (or digraph) D is: (1) *traceable* if D has a hamiltonian path; (2) *hamiltonian* if D has a hamiltonian cycle; (3) *strongly hamiltonian* if D has arcs and each arc lies on a hamiltonian cycle; (4) *hamiltonian-connected* if D has a hamiltonian u-v path for every pair of distinct vertices u and v; (5) *k-path traceable* if every path of length not exceeding k is contained in a hamiltonian path; and (6) *k-path hamiltonian* if every path of length not exceeding k is contained in a hamiltonian cycle.

The indegree and the outdegree of a vertex v are denoted by $\mathrm{id}(v)$ and $\mathrm{od}(v)$ respectively. A digraph D of order p is of *Ore-type* (k) if $\mathrm{od}(u) + \mathrm{id}(v) \geqslant p + k$ whenever u and v are distinct vertices for which uv is not an arc of D.

In this research announcement we outline a proof of the following result, a complete proof of which will appear elsewhere, and present some consequences of it.

THEOREM. *If a nontrivial digraph D is of Ore-type* (k), $k \geqslant 0$, *then D is k-path hamiltonian.*

PROOF. Let D have order $p \geqslant 2$. First, observe that D is strong. Since the result holds if D is the complete symmetric digraph K_p, we assume that $D \neq K_p$. This in turn implies that $p \geqslant k + 4$. Also, it can be shown that every path of length not exceeding k is contained in a path of length $(k + 1)$ and this longer path is contained in a cycle.

Suppose D has a path P: $v_1, v_2, \ldots, v_{k+1}$ of length k which is contained in no hamiltonian cycle. Let C: $v_1, v_2, \ldots, v_n, v_1$ be any longest cycle containing P. Then, $V \equiv V(D) - V(C) \neq \varnothing$, where $V(D)$ and $V(C)$ denote the vertex sets of D and C respectively.

Now, assume that V has distinct vertices u and v for which $uv \notin E(D)$ and the subdigraph $\langle V \rangle$ induced by V has no v-u path. Then, $vu \notin E(D)$ implies that

$$(1) \qquad p + k \leqslant \mathrm{od}(v) + \mathrm{id}(u) \leqslant p - n - 2 + \mathrm{od}(v, C) + \mathrm{id}(u, C)$$

where $\mathrm{od}(v, C)$ and $\mathrm{id}(u, C)$ denote the number of vertices in C which are

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dominated by v and dominate u, respectively. Then (1) implies that $n + k + 2 \leqslant \mathrm{od}(v, C) + \mathrm{id}(u, C)$ and this implies that $\langle V \rangle$ has no u-v path. For suppose that $\langle V \rangle$ has such a path. Since C is a longest cycle containing P, the digraph D cannot contain both of the arcs $v_i u$ and $v v_{i+1}$ for $k + 1 \leqslant i \leqslant n$. But this implies that $\mathrm{id}(u, C) + \mathrm{od}(v, C) \leqslant n + k$ and this is a contradiction. Using the fact that $uv \notin E(D)$, we obtain

$$p + k \leqslant \mathrm{od}(u) + \mathrm{id}(v) \leqslant p - n - 2 + \mathrm{od}(u, C) + \mathrm{id}(v, C)$$

which also implies that $n + k + 2 \leqslant \mathrm{od}(u, C) + \mathrm{id}(v, C)$. Together with the preceding result, this implies that either

$$n + k + 2 \leqslant \mathrm{od}(u, C) + \mathrm{id}(u, C) \quad \text{or} \quad n + k + 2 \leqslant \mathrm{od}(v, C) + \mathrm{id}(v, C).$$

In either case, it follows that D has a longer cycle containing P which is impossible. Thus, for distinct vertices u and v of $\langle V \rangle$, either $uv \in E(\langle V \rangle)$ or $\langle V \rangle$ has a v-u path. If $\langle V \rangle$ has a v-u path, then $\mathrm{od}(u, C) + \mathrm{id}(v, C) \leqslant n + k$. Thus,

$$\mathrm{od}(u, \langle V \rangle) + \mathrm{id}(v, \langle V \rangle) \geqslant p - n = |V|$$

whenever $u \neq v$ and $uv \notin E(\langle V \rangle)$. Hence, $\langle V \rangle$ is strongly connected.

Let W be the subpath $v_{k+1}, v_{k+2}, \ldots, v_n, v_{n+1} = v_1$ of C. Since $n \geqslant k + 2$, the path W has order at least 3; in fact W has at least 3 vertices which are dominated by vertices of V and at least 3 vertices which dominate vertices of V. It now suffices to consider the following two cases: (i) the path W has a nontrivial subpath W' whose initial vertex dominates a vertex of V and whose terminal vertex is dominated by a vertex of V; and (ii) the path W has no such subpath. Since consideration of either case leads to contradiction, our assumption that $V \neq \varnothing$ must be false. Hence, C is a hamiltonian cycle and the theorem follows.

Let $m, n \geqslant 1$ and $k \geqslant 0$. The symmetric join $K_{k+2} + (K_m \cup K_n)$ of K_{k+2} and the disjoint union of K_m and K_n is an Ore-type (k) digraph which is not $(k + 1)$-path hamiltonian. Hence, the result is "best possible."

The preceding result generalizes several results from graph theory and digraph theory, which we present below.

COROLLARY. *If the digraph D is of Ore-type (k), $k \geqslant -1$, then D is $(k + 1)$-path traceable.*

COROLLARY (WOODALL [5]). *If a nontrivial digraph is of Ore-type (0), then it is hamiltonian.*

COROLLARY. *If a nontrivial digraph is of Ore-type (1), then it is both strongly hamiltonian and hamiltonian-connected.*

A (undirected) graph of order p is of Ore-type (k) if the sum of the degrees of nonadjacent vertices is at least $(p + k)$. By considering symmetric digraphs, we obtain the following results.

COROLLARY (ORE [3]). *If a graph with order at least* 3 *is of Ore-type* (0), *then it is hamiltonian.*

COROLLARY (ORE [4]). *If a graph is of Ore-type* (1), *then it is hamiltonian-connected.*

COROLLARY (KRONK [2]). *If a graph of order* $p \geqslant 3$ *is of Ore-type* (k), $k \geqslant 0$, *then it is k-path hamiltonian.*

COROLLARY (KAPOOR AND THECKEDATH [1]). *If a graph is of Ore-type* (k), $k \geqslant -1$, *then it is* $(k + 1)$*-path traceable.*

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