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INVARIANCE PRINCIPLE FOR MODIFIED WAVE OPERATORS

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1. **Introduction.** The invariance principle of Birman and Kato (see e.g. [5]) states that, for simple scattering systems with short-range potentials, the wave operator limits

$$\Omega_\pm(\phi(H_2), \phi(H_1)) \equiv \operatorname*{s-lim}_{t \to \pm\infty} e^{it\phi(H_2)} e^{-it\phi(H_1)} P_1 \tag{1}$$

are independent of ϕ for a wide class of functions, and equal, respectively, the wave operators

$$\Omega_\pm(H_2, H_1) \equiv \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_2} e^{-itH_1} P_1. \tag{2}$$

Kato first proved the invariance under the assumption that $H_2 - H_1$ is a trace-class operator. It has since been proved under some alternative assumptions on H_1 and H_2 (see e.g. [4], [6]).

For other scattering systems, such as scattering with long-range potentials, the limits (1) and (2) may not exist; however, certain modified wave operators (see (3) and (4) below) may exist [1], [3]. An invariance principle for modified wave operators has been proved by Matveev [6], [7] and Sakhnovich [8] under certain rate-of-convergence assumptions. However, these assumptions are shown to be satisfied only for a class of short-range potentials [6, Theorem 2].

In this note we announce the result that the invariance principle of scattering theory is valid in practically all situations in which (possibly modified) time-dependent wave operators are known to exist.

2. **Notation.** Let H_k be selfadjoint operators on separable Hilbert spaces $\mathcal{H}_k$, $k = 1, 2$. Let P_1 denote the orthogonal projection of $\mathcal{H}_1$ onto the space $\mathcal{H}_{1,ac}$ of absolute continuity for H_1. Let Δ be some closed and bounded interval of the real axis $\mathbf{R}$, and let $E_1(\Delta)$ be the corresponding spectral projection of the operator H_1. Let $\mathcal{D}_1$ be the dense subset of vectors $u \in \mathcal{H}_{1,ac}$ with $|||u||| < \infty$,

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where $|||u|||$ is defined as in [5, p. 542]. Let J be a bounded identification operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ which maps the domain of H_1 into the domain of H_2.

Let $U(t)$ be a uniformly bounded operator-valued function of t, which commutes with H_1 for all t, and for $|t|$ sufficiently large is invertible and satisfies

$$\operatorname*{s-lim}_{t\to\pm\infty} U^{-1}(t)U(t+s) = I$$

for all s. For example, $U(t)$ may be the long-range modification operators introduced in [1], [3].

Local modified wave operators $W_\pm^\Delta$ are defined by

$$W_\pm^\Delta \equiv \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_2}Je^{-itH_1}U(t)E_1(\Delta)P_1, \tag{3}$$

whenever these limits exist.

DEFINITION. The real-valued function ϕ is said to be a *Kato function* if $\mathbf{R}$ can be divided into a finite number of subintervals in such a way that, in each open subinterval, ϕ is differentiable with ϕ' continuous, locally of bounded variation, and positive [5, Lemma X–4.6].

Let

$$Q_\phi(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}(t, s)e^{-isH_1}U(s)ds,$$

where $\hat{R}(t, s) \equiv \int_{-\infty}^{\infty} e^{is\eta - it\phi(\eta)}K_\Delta(\eta)d\eta$, and $K_\Delta(\eta)$ is a smooth real-valued function of compact support equal to unity on Δ.

Define

$$W_\pm^\Delta(\phi(H_2), \phi(H_1); Q_\phi) \equiv \operatorname*{s-lim}_{t\to\pm\infty} e^{it\phi(H_2)}JQ_\phi(t)E_1(\Delta)P_1. \tag{4}$$

3. **Main results.**

THEOREM (INVARIANCE PRINCIPLE). *Let ϕ be a Kato function, and suppose there is a dense subset $\mathcal{D}$ of $E_1(\Delta)\mathcal{H}_{1,\mathrm{ac}} \cap \mathcal{D}_1$ with the property that for every $u \in \mathcal{D}$ there exists a $\tau > 0$ such that*

$$L(t)u \equiv [(H_2J - JH_1)e^{-itH_1}U(t) - Je^{-itH_1}U'(t)]u$$

is defined and strongly continuous on $\mathbf{R}_\tau \equiv \{t \in \mathbf{R}\colon |t| > \tau\}$, and

$$\|L(t)u\| = O(|t|^{-1-\epsilon}) \quad \textit{as } t \to \pm\infty \textit{ for some } \epsilon > 0. \tag{5}$$

Then the wave operators $W_\pm^\Delta$ and $W_\pm^\Delta(\phi(H_2), \phi(H_1); Q_\phi)$ exist, and are, respectively, equal.

REMARK. The closed bounded interval Δ assures that the integral defining $Q_\phi(t)$ converges. In situations, such as short-range scattering, where $Q_\phi(t)$ does not depend on Δ, then a global version ($E_1(\Delta) = I$) of our theorem is valid.

The proof of the above theorem relies upon the following two lemmas.

LEMMA 1. *Suppose that the Hilbert-space-valued function* $\hat{h}(s)$ *and its strong derivative* $\hat{h}'(s)$ *are strongly continuous and satisfy*:

(i) $\|\hat{h}(s)\| \longrightarrow 0$ *as* $|s| \longrightarrow \infty$,

(ii) $\|\hat{h}'(s)\| \in L_1(\mathbf{R}) \cap L_2(\mathbf{R})$, *and*

(iii) $|s|^\alpha \|\hat{h}'(s)\| \in L_1(\mathbf{R})$ *for some* α $(0 < \alpha \leqslant 1)$.

Then $\hat{h}(s)$ *is the Fourier transform of a Bochner integrable function.*

LEMMA 2. *If* ϕ *is a Kato function and* $\hat{h}(s)$ *is the strong Fourier transform of a Bochner integrable function, then*

$$\operatorname*{s-lim}_{t \longrightarrow \pm\infty} \int_{-\infty}^{\infty} \hat{R}(t, s) e^{-isH_2} \hat{h}(s) ds = 0.$$

4. **Consequences.** Existence of time-dependent wave operators is normally established by proving an estimate of the form (5) for a particular choice of H_1, H_2, J, and $U(t)$. Our theorem gives the invariance in all of these situations. These include, for example, the cases of single or multi-channel scattering with either short or long-range potentials, classical scattering, certain relativistic scattering, and even scattering for certain rapidly oscillating potentials which are possibly unbounded at infinity.

More general results, proofs, and applications will appear elsewhere [2].

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