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THE Q-MATRIX PROBLEM FOR MARKOV CHAINS

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1. Let I be a countable set. A ("standard") *Markov transition function* $(P(t))$ on I may be regarded as a family $\{p_{ij}(\cdot): i, j \in I\}$ of functions on $[0, \infty)$ such that (for $i, j \in I$ and $s, t \in [0, \infty)$)

$$p_{ij}(t) \geqslant 0, \quad \sum_{k \in I} p_{ik}(t) = 1,$$

$$p_{ij}(s+t) = \sum_{k \in I} p_{ik}(s) p_{kj}(t), \quad \lim_{u \downarrow 0} p_{ii}(u) = p_{ii}(0) = 1.$$

If $(P(t))$ is a Markov transition function on I, the (Doob-Kolmogorov) limits

$$-q_{ii} = \lim_{t \downarrow 0} t^{-1}[1 - p_{ii}(t)], \quad q_{ij} = \lim_{t \downarrow 0} t^{-1} p_{ij}(t)$$

exist in $[0, \infty]$ and satisfy

(DK1) $$0 \leqslant q_{ij} < \infty \quad (i \neq j),$$

(DK2) $$\sum_{k \neq i} q_{ik} \leqslant -q_{ii} \leqslant \infty.$$

The $I \times I$ matrix $Q = (q_{ij})$ is called the *Q-matrix* of $(P(t))$ and we write $Q = P'(0)$.

The following theorem solves the *Q-matrix problem* for the case when all states are instantaneous ($q_{ii} = -\infty$, $\forall i$).

THEOREM. *Let Q be an $I \times I$ matrix with*

(1) $$q_{ii} = -\infty \quad (\forall i); \qquad 0 \leqslant q_{ij} < \infty \quad (\forall i, j: i \neq j).$$

For Q to be the Q-matrix of a Markov transition function $(P(t))$, it is necessary and sufficient that the following conditions (2) *and* (3) *hold*:

(2) $$\sum_{j \notin \{a,b\}} q_{aj} \wedge q_{bj} < \infty \quad (\forall a, b: a \neq b);$$

(3) *for every finite subset H of I,*

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$$\liminf_{j} \sum_{h \in H} q_{hj} = 0.$$

[*Note.* The theorem remains true if "Markov" is replaced by "sub-Markov".]

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2. **Sketched proof of necessity of** (2) **and** (3). Suppose that Q is the Q-matrix of a Markov transition function $(P(t))$. Let X be a Ray version of a Markov process (taking values in some Ray-Knight compactification $\bar{E}$ of I) with Ray transition function extending $(P(t))$. See Getoor [2].

LEMMA 1 (LOCAL CHARACTER PROPERTY). *Let G be an open subset of $\bar{E}$ and let $h \in G \cap I$. Then (with G^c denoting $\bar{E}\backslash G$),*

$$\sum_{j \in G^c \cap I} q_{hj} < \infty. \tag{4}$$

Notation. Write $Q(h, G^c \cap I)$ for the sum (4).

PROOF. Define

$$X_t^G = X_t, \qquad t < T_{G^c} = \inf\{s > 0: X_s \in G^c\},$$

$$= \partial, \qquad t \geqslant T_{G^c}.$$

Then X^G is a "standard" chain with "minimal" state-space $(G \cap I) \cup \partial$. By (DK1),

$$\infty > \lim_{t \downarrow 0} t^{-1} P^h\{X_t^G = \partial\} \geqslant Q(h, G^c \cap I).$$

Condition (2) now follows from the Hausdorff property of the (metrizable) Ray-Knight topology. Condition (3) follows from the fact that under hypothesis (1) no point of I can be isolated in I.

Notes. (i) The above arguments modernise those used to prove (2) and (3) in my 1967 paper [4].

(ii) Property (3) may be expressed in the following equivalent form:

(3*) there exists an *infinite* subset K of I such that $Q(i, K\backslash i) < \infty$, $\forall i \in I$.

3. Proof of the *sufficiency* of conditions (2) and (3) is much harder. The main motivation comes from the probabilistic "branching" procedure which underlies Kendall's remarkable analytic construction of the "tree" process in the 1958 paper [3]. However, the "local character property" dominates the proof of "sufficiency" too.

We now assume that Q is an $I \times I$ matrix satisfying (1), (2) *and* (3).

We say that I is tree-labelled if I is "labelled": $I = I_0 \cup I_1 \cup I_2 \cup \cdots$, where $I_0 = \{0\}$, and, for $n \in \mathbf{N}$,

$$I_n = \{0 i_1 i_2 \cdots i_n: i_1, i_2, \ldots, i_n \in \mathbf{N}\}.$$

Let π be the *immediate predecessor* map of $I\backslash\{0\}$ to I:

$$\pi i = 0i_1 i_2 \cdots i_{n-1} \in I_{n-1} \quad \text{when } i = 0i_1 i_2 \cdots i_n \in I_n.$$

We can regard π as *being* (rather than "inducing") a tree-labelling.

LEMMA 2 (P. D. SEYMOUR). *There is a tree-labelling π of I such that*

4(i) $$\liminf_{j\in\pi^{-1}\{i\}} q_{ij} = 0,$$

4(ii) $$\sum_{j\neq i} [q_{ij} - q_{ij}^-] < \infty, \quad \forall i \in I,$$

where Q^- is the $I \times I$ matrix defined as follows:

$$q_{ij}^- = q_{ij} \quad \textit{if } j \in \{i\} \cup \pi^{-1}\{i\},$$
$$= 0 \quad \textit{otherwise}.$$

We now assume I is already tree-labelled in accordance with Lemma 2.

4. For each i, we choose a certain special type of chain $X^{(i)}$ with minimal state-space $\{i\} \cup \pi^{-1}\{i\}$, with Markov transition function, and with Q-matrix $Q^{(i)}$ satisfying

5(i) $$-q_{ii}^{(i)} = \infty, \quad q_{ij}^{(i)} = q_{ij} \qquad (j \in \pi^{-1}\{i\}),$$

5(ii) $$-q_{jj}^{(i)} < \infty, \quad q_{jk}^{(i)} = 0 \qquad (j \in \pi^{-1}\{i\}, k \neq j).$$

It is important to realise that condition 4(i) is both *necessary and sufficient* for the existence of a chain $X^{(i)}$ with Markov transition function and with Q-matrix satisfying conditions (5). It should also be realised that $X^{(i)}$ is necessarily a complicated chain with infinitely many fictitious states.

5. **Kendall's branching procedure.** The chain $X^{(0)}$ has minimal state-space $\{0, 01, 02, \ldots\}$. The state 01 is stable for $X^{(0)}$ with rate a_{01} (say). During each (exponentially distributed) visit by $X^{(0)}$ to 01, replace $X^{(0)}$ by a chain on $\{01\} \cup \pi^{-1}\{01\}$ with the $P^{(01)}$ law of $X^{(01)}$. (In effect, this last-mentioned chain is killed at rate a_{01}.) After modifying $X^{(0)}$ on its "01-intervals", we obtain a chain on $\{0, 01\} \cup \{02, 03, \ldots\} \cup \{011, 012, \ldots\}$ for which the first two states (and only these) are instantaneous.

By proceeding in the obvious inductive fashion, we can—*provided that we have chosen the $X^{(i)}$ with sufficient care*—obtain a "projective limit" chain X^- with Q-matrix Q^-. Since Q differs only "finitely" from Q^- in the sense of 4(ii), it is easy (with the same proviso as above) to extend the Levy system of X^- so as to produce a chain X with the desired Q-matrix Q.

The way to choose the $X^{(i)}$ will be explained in the "full" version of this work. The existence of the limit chain X^- is proved by the method in Freedman's book [1].

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