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DUALITY FOR CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS BY LOCALLY COMPACT GROUPS

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The duality for crossed products of von Neumann algebras by locally compact abelian groups has been obtained by Takesaki [4]. We shall generalize this result to a locally compact (not necessarily abelian) group by using the Fourier algebra in place of the dual group.

Let G denote a locally compact group with a right invariant Haar measure dt, and M denote a von Neumann algebra over a Hilbert space $\mathcal{H}$. By an *action* of G on M we mean a homomorphism σ: $t \in G \mapsto \sigma_t \in \mathrm{Aut}(M)$ such that for each $x \in M$ the mapping $t \in G \mapsto \sigma_t(x)$ is σ-strongly* continuous. Let $\{\pi_\sigma, \lambda\}$ be a covariant representation of $\{M, \sigma\}$ on $\mathcal{H} \otimes L^2(G)$ defined by

$$\begin{cases} (\pi_\sigma(x)\xi)(s) \equiv \sigma_s(x)\xi(s), & \xi \in \mathcal{H} \otimes L^2(G), \\ \lambda(r)\xi(s) \equiv \xi(sr), & r, s \in G. \end{cases}$$

Then the crossed product $\mathcal{R}(M; \pi_\sigma)$ of M by G is the von Neumann algebra generated by $\pi_\sigma(M)$ and $\lambda(G)$.

THEOREM 1. *A necessary and sufficient condition that a mapping α of M into $M \otimes L^\infty(G)$ be induced by an action σ with*

$$(\alpha(x)\xi)(s) = \sigma_s(x)\xi(s), \qquad x \in M, \xi \in \mathcal{H} \otimes L^2(G),$$

is that α be an isomorphism with the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\quad\alpha\quad} & M \otimes L^\infty(G) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha \otimes \iota} \\ M \otimes L^\infty(G) & \xrightarrow{\iota \otimes \delta} & M \otimes L^\infty(G) \otimes L^\infty(G), \end{array} \tag{1}$$

where $(\delta f)(s, t) \equiv f(st)$ for $f \in L^\infty(G)$.

For the right regular representation λ_G of G on $L^2(G)$, i.e.,

$$(\lambda_G(s)f)(t) \equiv f(ts), \qquad f \in L^2(G), s, t \in G,$$

let $R(G)$ denote the von Neumann algebra generated by $\lambda_G(G)$. Let γ denote the isomorphism of $R(G)$ into $R(G) \otimes R(G)$ defined by

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$$\gamma(\lambda_G(s)) \equiv \lambda_G(s) \otimes \lambda_G(s), \quad s \in G.$$

DEFINITION. For an isomorphism β of a von Neumann algebra N into $N \otimes R(G)$ with the commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\quad\beta\quad} & N \otimes R(G) \\ \beta\big\downarrow & & \big\downarrow \beta \otimes \iota \\ N \otimes R(G) & \xrightarrow{\iota \otimes \gamma} & N \otimes R(G) \otimes R(G), \end{array} \tag{2}$$

we define a *crossed dual product* of N by G as the von Neumann algebra generated by $\beta(N)$ and $1 \otimes L^\infty(G)$. We denote it by $\mathcal{R}_d(N; \beta)$.

THEOREM 2. *Let W and V be unitaries on $\mathcal{H} \otimes L^2(G) \otimes L^2(G)$ defined by*

$$(W\xi)(s, t) \equiv \xi(s, ts) \quad \textit{and} \quad (V\xi)(s, t) \equiv \xi(st, t).$$

If α (resp. β) is an isomorphism of M (resp. N) into $M \otimes L^\infty(G)$ (resp. $N \otimes R(G)$) with the commutative diagram (1) (resp. (2)), then $\hat{\alpha}$ (resp. $\hat{\beta}$) defined by

$$\hat{\alpha}(y) \equiv W^*(y \otimes 1)W \quad (\textit{resp. } \hat{\beta}(z) \equiv V(z \otimes 1)V^*)$$

is an isomorphism of $\mathcal{R}(M; \alpha)$ (resp. $\mathcal{R}_d(N; \beta)$) into $\mathcal{R}(M; \alpha) \otimes R(G)$ (resp. $\mathcal{R}_d(N; \beta) \otimes L^\infty(G)$) with the commutative diagram (2) for $\mathcal{R}(M; \alpha)$ and $\hat{\alpha}$ (resp. (1) for $\mathcal{R}_d(N; \beta)$ and $\hat{\beta}$).

Making use of the above two theorems we can give the following duality theorem for crossed products of von Neumann algebras by locally compact groups. When G is abelian, its corollary is nothing but a duality theorem of Takesaki [4].

THEOREM 3 (DUALITY). *Under the notations in Theorem 2, let σ be an action of G on M, $\alpha \equiv \pi_\sigma$, $\beta \equiv \hat{\alpha}$, $\tilde{\alpha} \equiv \hat{\beta}$ and $\tilde{\sigma}$ the action associated with $\tilde{\alpha}$ as in Theorem 1. Let π be a faithful representation of M on $\mathcal{H} \otimes L^2(G) \otimes L^2(G)$ such that*

$$(\pi(x)\xi)(s, t) = \sigma_{st^{-1}}(x)\xi(s, t),$$

and let Λ_1 and Λ_2 be a representation and a unitary representation of G on $\mathcal{H} \otimes L^2(G) \otimes L^2(G)$ defined by

$$(\Lambda_1(r)\xi)(s, t) \equiv \xi(s, r^{-1}t) \quad \textit{and} \quad (\Lambda_2(r)\xi)(s, t) \equiv \xi(s, tr),$$

respectively. Then $\mathcal{R}_d(\mathcal{R}(M; \alpha); \beta)$ is isomorphic to $\pi(M) \otimes B(L^2(G))$ and the isomorphism transforms the action σ of G on the former into the action of G on the latter given by $\mathrm{Ad}(\Lambda_2(r)) \otimes \mathrm{Ad}(\lambda_G(r))$ for $r \in G$. In particular,

$$\pi(\sigma_r(x)) = \Lambda_1(r)\pi(x)\Lambda_1(r)^{-1} \quad \textit{and} \quad \tilde{\sigma}_r(\pi(x)) = \Lambda_2(r)\pi(x)\Lambda_2(r)^{-1}.$$

When G is unimodular, we can define a unitary U on $\mathcal{H} \otimes L^2(G) \otimes L^2(G)$ by

$$(U\xi)(s, t) \equiv \Delta(t)^{1/2}\xi(t^{-1}s, t),$$

and a mapping $\hat{\beta}'$ of $R_d(N;\beta)$ into $R_d(N;\beta)\otimes L^\infty(G)$ by

$$\hat{\beta}'(z) \equiv U(z\otimes 1)U^*.$$

Then $\hat{\beta}'$ is an isomorphism which makes commutative the diagram (1) for $R_d(N;\beta)$ and $\hat{\beta}'$.

COROLLARY. *Assume that G is unimodular. Under the notations in Theorem 2, let σ be an action of G on M, $\alpha \equiv \pi_\sigma$, $\beta \equiv \hat{\alpha}$, $\tilde{\alpha} \equiv \hat{\beta}'$ and $\tilde{\sigma}$ the action associated with $\tilde{\alpha}$ as in Theorem 1. Then $R_d(R(M;\alpha);\beta)$ is isomorphic to $M \otimes B(L^2(G))$ and the isomorphism transforms the action $\tilde{\sigma}$ of G on the former into the action of G on the latter given by $\sigma_r \otimes \mathrm{Ad}(\lambda'_G(r))$ for $r \in G$, where λ'_G is the left regular representation of G on $L^2(G)$.*

THEOREM 4 (DUALITY). *Under the notations in Theorem 2, $R(R_d(N;\beta);\alpha)$ is isomorphic to $N \otimes B(L^2(G))$.*

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REFERENCES

1. G. W. Mackey, *A theorem of Stone and von Neumann,* Duke Math. J. 16 (1949), 313–326. MR 11, 10.

2. A. Ikunishi and Y. Nakagami, *On invariants $G(\sigma)$ and $\Gamma(\sigma)$ for an automorphism group of a von Neumann algebra,* Publ. RIMS, Kyoto, Univ. (to appear).

3. M. Takesaki, *Duality and von Neumann algebras,* Lectures on Operator Algebras (K. H. Hofmann, editor), Lecture Notes in Math., vol. 247, Springer-Verlag, Berlin, 1972, pp. 666–779.

4. ———, *Duality for crossed products and the structure of von Neumann algebras of type* III, Acta Math. 131 (1973), 249–310.

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