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THE STRUCTURE OF SINGULARITIES IN AREA-RELATED VARIATIONAL PROBLEMS WITH CONSTRAINTS

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This is a research announcement of results whose full details and proofs have been submitted for publication elsewhere. We provide a complete description, both combinatorial and differential, of the local structure of singularities in a large class of two-dimensional surfaces in $\mathbf{R}^3$, those which are $(\mathbf{M}, \epsilon, \delta)$ minimal **[TJ1]** and those which are $(\mathbf{F}, \epsilon, \delta)$ minimal for a Hölder continuous ellipsoidal integrand F **[TJ2]**. Such surfaces include mathematical models for compound soap bubbles **[AF1]**, **[AF2]** and soap films, thereby settling a problem which has been studied for well over a century (a very general formulation of Plateau's Problem); in general, $(\mathbf{M}, \epsilon, \delta)$ and $(\mathbf{F}, \epsilon, \delta)$ minimal surfaces arise as solutions to geometric variational problems with constraints.

$(\mathbf{M}, \epsilon, \delta)$ and $(\mathbf{F}, \epsilon, \delta)$ minimal surfaces were defined, shown to exist, and proven to be regular almost everywhere in **[AF2]** (see **[AF1]** for a brief description). We define $Y \subset \mathbf{R}^3$ as the union of the half disk $\{x \in \mathbf{R}^3: x_1^2 + x_2^2 \leqslant 1, x_2 \geqslant 0, x_3 = 0\}$ with its rotations by 120° and 240° about the x_1 axis, and define $T \subset \mathbf{R}^3$ as $C \cap \{x: |x| \leqslant 1\}$, where C is the central cone over the one-skeleton of the regular tetrahedron centered at the origin and containing as vertices the points $(3, 0, 0)$ and $(-1, 2\sqrt{2}, 0)$. Varifold tangents are defined in **[AW 3.4]** and a tangent cone is defined to be the support of a varifold tangent.

The major result of **[TJ1]** is the following.

THEOREM. *Suppose S is $(\mathbf{M}, \epsilon, \delta)$ minimal with respect to some closed set B, where $\epsilon(r) = Cr^\alpha$ for some $C < \infty$ and $\alpha > 0$. Then*

(1) *there exists a unique tangent cone, denoted* $\mathrm{Tan}(S, p)$, *to S at each point p in S,*

(2) $R(S) = \{p \in S$: $\mathrm{Tan}(S, p)$ *is a disk*$\}$ *is a two-dimensional Hölder continuously differentiable submanifold of* $\mathbf{R}^3$, *with* $\mathcal{H}^2(R(S)) = \mathcal{H}^2(S)$ **[AF1]**, **[AF2]** (*here $\mathcal{H}^2$ denotes (Hausdorff) two-dimensional area*),

(3) $\sigma_Y(S) = \{p \in S$: $\mathrm{Tan}(S, p) = \theta Y$ *for some θ in* $\mathbf{O}(3)$, *the group of*

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orthogonal rotations of $\mathbf{R}^3$} *is a one-dimensional Hölder continuously differentiable submanifold of* $\mathbf{R}^3$, *and for each* p *in* $\sigma_Y(S)$ *there exists a neighborhood* N *of* p *and a Hölder continuously differentiable diffeomorphism* $f: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ *such that* $f(S \cap N) = Y$,

(4) $\sigma_T(S) = \{p \in S: \operatorname{Tan}(S, p) = \theta T$ *for some* θ *in* $\mathbf{O}(3)\}$ *consists of isolated points, and for each* p *in* $\sigma_T(S)$ *there is a neighborhood* N *of* p *and a Hölder continuously differentiable diffeomorphism* $f: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ *such that* $f(S \cap N) = T$,

(5) $S = R(S) \cup \sigma_Y(T) \cup \sigma_T(S)$.

The proof uses the methods of geometric measure theory, in particular those developed in **[AF2]** and **[TJ3]**, and can be divided into three broad regions: a tangent cone analysis, the proof of an "epiperimetric inequality" similar to those of **[R]** and **[TJ3]** but in derivative form, and the derivation of the differential structure from this inequality. Contained in the first region is a proof that the only two-dimensional area minimizing cones in $\mathbf{R}^3$ are (up to rotations) the disk Y and T; Lamarle **[L]** in 1864 attempted to do this, but his analysis was partly in error.

Recall that an integrand is a continuous function $F: \mathbf{R}^3 \times \mathbf{G}(3, 2) \longrightarrow \mathbf{R}^+$, where $\mathbf{G}(3, 2)$ denotes the Grassmannian of unoriented two-plane directions in $\mathbf{R}^3$, and that at each point p in $\mathbf{R}^3$ there is associated to F the constant coefficient integrand $F^p: \mathbf{G}(3, 2) \longrightarrow \mathbf{R}^+$ given by $F^p(\pi) = F(p, \pi)$ for every π in $\mathbf{G}(3, 2)$ **[AF2]**. F^p may be regarded as a function on the unit ball in $\bigwedge_2 \mathbf{R}^3$, the space of 2-vectors of $\mathbf{R}^3$, by defining $F^p(v) = F^p(\pi)$, where v is any 2-vector of length 1 and π is the unoriented two-plane naturally associated to v **[F1, 6.1]**. We define F to be *ellipsoidal* if for each p in $\mathbf{R}^3$ the positively homogeneous function of degree one on $\bigwedge_2 \mathbf{R}^3$ which extends F^p is a norm induced by an inner product on $\bigwedge_2 \mathbf{R}^3$. Equivalently, for every p in $\mathbf{R}^3$ there exists a nonsingular linear map $L_p: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that $\mathbf{F}^p(v) = |\bigwedge_2 L_p(v)|$ for every 2-vector v of length 1; thus for every $(\mathcal{H}^2, 2)$-rectifiable set S, $\mathbf{F}^p(S) = \mathcal{H}^2(L_p(S))$.

The major result of **[TJ2]** is then that if S is $(\mathbf{F}, \epsilon, \delta)$ minimal with respect to some closed set B for a Hölder continuous ellipsoidal integrand F (and $\epsilon(r) = Cr^\alpha$ for some $C < \infty$ and $\alpha > 0$), the conclusions of the above theorem hold, except that "θY" is replaced by "$L_p^{-1}(\theta Y)$" and "θT" by "$L_p^{-1}(\theta T)$". This implies in particular that area minimizing surfaces (more generally, $(\mathbf{M}, \epsilon, \delta)$ minimal surfaces) on three-dimensional Hölder continuously differentiable manifolds have the structure described in the theorem; such a result could be derived directly from **[TJ1]** only if the manifold were at least C^2.

A converse of these results is easy to prove, i.e., if a compact surface has the structure of (1)–(5) of the theorem [respectively, has that structure up to a linear map at each point], then the surface is $(\mathbf{M}, \epsilon, \delta)$ minimal [respectively, $(\mathbf{F}, \epsilon, \delta)$ minimal for some Hölder continuous ellipsoidal integrand F] for some $\delta > 0$ and $\epsilon(r) = Cr^\alpha$, some $C < \infty$ and $\alpha > 0$.

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