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CLASSIFICATION OF AUTOMORPHISMS OF HYPERFINITE FACTORS OF TYPE II_1 AND II_∞ AND APPLICATION TO TYPE III FACTORS

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ABSTRACT. For each integer $p = 0, 1, 2 \cdots$ and complex number γ, $\gamma^p = 1$ ($\gamma = 1$ for $p = 0$) we define an automorphism s_p^γ of the hyperfinite factor of type II_1, R. For any automorphism α of R there is a unique couple (p, γ) and a unitary $v \in R$ such that α is conjugate to $\mathrm{Ad}\, v \circ s_p^\gamma$. Let $R_{0,1}$ be the tensor product of R by a I_∞ factor. There is, up to conjugacy, only one automorphism θ_λ of $R_{0,1}$ such that θ_λ multiplies the trace by λ, provided $\lambda \neq 1$.

Introduction. The classification of type III factors that we proposed in [2] relates isomorphism classes of type III_λ factors, $\lambda \in]0, 1[$ with outer conjugacy classes of automorphisms of factors of type II_∞. An obvious criticism to the value of such a relation is then the following: Is it possible to classify automorphisms even for the simplest factor of type II_∞, namely $R_{0,1}$ the tensor product of R, the hyperfinite II_1, by a I_∞ factor. We answer this question in this paper, showing that for any $\lambda \in]0, 1[$ there is only one automorphism, up to conjugacy, of $R_{0,1}$ which multiplies the trace by λ. The proof of this fact relies on the classification of automorphisms of the hyperfinite factor R (see Theorem 1) which in turn uses mainly the analogy between classical ergodic theory and ergodic theory on a nonabelian von Neumann algebra.

Automorphisms of the hyperfinite factor of type II_1. Recall that if M is a factor and $\theta \in \mathrm{Aut}\, M$, one defines the outer period $p_0(\theta)$ as the period of θ modulo inner automorphisms (i.e. $\theta^k \in \mathrm{Int}\, M \Leftrightarrow k \in p_0(\theta)Z$). Also the obstruction of θ, noted $\gamma(\theta)$, is the root of unity in **C** such that ($\theta^{p_0} = \mathrm{Ad}\, v$, v unitary in M) $\Rightarrow \theta(v) = \gamma v$. Finally α and $\beta \in \mathrm{Aut}\, M$ are outer conjugate iff β is conjugate to the product of α by an inner automorphism.

THEOREM 1. *Two automorphisms α, β of R are outer conjugate if and only if $p_0(\alpha) = p_0(\beta)$ and $\gamma(\alpha) = \gamma(\beta)$.*

In particular, any two aperiodic automorphisms α, β of R are outer conjugate. This relies on an analogue of Rokhlin's theorem. In the case $p_0(\alpha) \neq 0$ the proof uses the tensor product as a group structure on the set $Br(Z/p, R)$ of

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outer conjugacy classes of α's with $p_0(\alpha) = p$ (see [5]). For $p \neq 0$ and $\gamma \in \mathbf{C}$, $\gamma^p = 1$, there is, up to conjugacy, only one automorphism s_p^γ of R with period p order γ and invariants p, γ. This automorphism s_p^γ has been described in [4], [5]. For $p = 0$ we let s_0 be the bilateral shift on R when R is written $R = \bigotimes_{\nu \in Z}(R_1)_\nu$ with R_1 isomorphic to R.

Theorem 1 means that up to conjugacy any automorphism of R is the product of an s_p^γ by an inner automorphism.

COROLLARY 2. *The group* Out R = Aut $R/\mathrm{Int}\,R$ *has no nontrivial normal subgroup.*

In particular the center of Out R is trivial, unlike for any type III factors [2, 1.2.8b].

Corollary 2 means that R cannot break in a significant and invariant way into simpler objects.

COROLLARY 3. *Let N be a finite von Neumann algebra generated by a hyperfinite von Neumann subalgebra P and a unitary U, $UPU^* = P$, then N is hyperfinite.*

One shows, using Theorem 1, that any automorphism α of P generates the same full group [2, 1.5.4] as an automorphism β such that for some increasing sequence of finite dimensional subalgebras K_ν, $\nu \in N$, of P one has $\beta(K_\nu) = K_\nu$ for all ν, and $\bigcup_{\nu=1}^\infty K_\nu$ dense in P. Then one replaces U, Ad $U/P = \alpha$ by a unitary $V \in N$ such that Ad $V/P = \beta$.

With Corollary 3 one can then prove a result, due to Golodets when N is properly infinite (see [6]).

COROLLARY 4. *Let P be a hyperfinite von Neumann algebra and G a solvable group of unitaries in $L(H)$ such that $vPv^* = P$ for all $v \in G$; then $(P \cup G)''$ is hyperfinite.*

In particular, any representation of a solvable group generates a hyperfinite von Neumann algebra.

Automorphisms of the known hyperfinite factor of type II_∞. Let N be a factor of type II_∞, τ a faithful semifinite normal trace on N, $\theta \in$ Aut N; then we call the unique $\lambda \in R_+^*$ such that $\tau \circ \theta = \lambda\tau$ the module of θ: $\lambda = \mathrm{mod}\,\theta$.

THEOREM 5. *Let $R_{0,1} = R \otimes L(H)$ be the known hyperfinite factor of type II_∞. For any $\lambda \in R_+^*$, $\lambda \neq 1$, there is up to conjugacy only one $\theta \in$ Aut $R_{0,1}$ with module equal to λ.*

Also, two automorphisms α, β or $R_{0,1}$ with module equal to 1 are outer conjugate iff they have the same outer period and the same obstruction.

COROLLARY 6. *An automorphism $\theta \in$ Aut $R_{0,1}$ is a commutator $\theta =$*

$\alpha\beta\alpha^{-1}\beta^{-1}$ *for* $\alpha, \beta \in \mathrm{Aut}\, R_{0,1}$ *if and only if its module is equal to* 1.

COROLLARY 7. *Let* M *be a factor of type* III_λ *and* $M = W^*(\theta, N)$ *its discrete decomposition* [2, Theorem 4.4.1]. *Then* M *is isomorphic to Powers factor* R_λ *iff* N *is isomorphic to* $R_{0,1}$.

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