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THE APPROACH OF SOLUTIONS OF NONLINEAR DIFFUSION EQUATIONS TO TRAVELLING WAVE SOLUTIONS

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1. This note is concerned with the pure initial-value problem for the nonlinear diffusion equation

$$(1) \qquad u_t - u_{xx} - f(u) = 0 \qquad (-\infty < x < \infty,\ t > 0),$$

with $u(x, 0) = \phi(x)$. This problem has attracted an increasing amount of attention in recent years, one of the central questions being whether or not the solution $u(x, t)$ tends as $t \longrightarrow \infty$ to a travelling wave solution $U(x - ct)$. ([1] gives a general bibliography.) We adopt the usual normalization of the problem by assuming throughout that $f \in C^1[0, 1]$, $f(0) = f(1) = 0$, $0 \leqslant \phi \leqslant 1$, so that, as is well known, $0 \leqslant u(x, t) \leqslant 1$ for all x, t.

2. A typical convergence result that we can prove is the following.

THEOREM A. *Let $f \in C^1[0, 1]$, with $f(0) = f(1) = 0, f'(0) < 0, f(1) < 0$,*

$$f(u) < 0 \quad for\ 0 < u < \alpha_0, \qquad f(u) > 0 \quad for\ \alpha_1 < u < 1,$$

and assume that there exists a travelling wave solution $U(x - ct)$ with $U(-\infty) = 1$, $U(\infty) = 0$, $0 \leqslant U \leqslant 1$. Let ϕ satisfy $0 \leqslant \phi \leqslant 1$, $\liminf_{x \to -\infty} \phi(x) > \alpha_1$, $\limsup_{x \to \infty} \phi(x) < \alpha_0$. Then there exists some x_0 such that,

$$\lim_{t \to \infty} |u(x, t) - U(x - ct - x_0)| = 0$$

uniformly in x. If ϕ is monotonic, then the approach is in fact exponential.

We remark that such a travelling wave U can be shown to be necessarily monotonic, and it is an obvious consequence of Theorem A that U is unique up to translation. This can, of course, be shown directly (Theorem C below), and conditions under which U will exist are discussed in Theorem D.

In some cases the solution develops into a pair of diverging travelling waves, and this is relevent to the case where ϕ is of compact support.

THEOREM B. *Let f satisfy the hypotheses of Theorem* A, *and suppose* $c >$

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0. *Let* ϕ *satisfy* $0 \leqslant \phi \leqslant 1$, $\limsup_{x \to \pm\infty} \phi(x) < \alpha_0$, *and* $\phi(x) > \alpha_1 + \eta$ *for some* $\eta > 0$ *and a sufficiently long x-interval. Then there exists some* x_0 *and* x_1 *such that, uniformly in x,*

$$\lim_{t \to \infty} |u(x, t) - U(x - ct - x_0) - U(-x - ct - x_1) + 1| = 0.$$

We remark that the condition $c > 0$ is equivalent to $\int_0^1 f\,du > 0$.

To prove Theorems A and B, we write the solution as a function $u = u^*(z, t)$, where $z = x - ct$. A comparison technique based on the maximum principle is used to obtain information about u^* as $z, t \longrightarrow \infty$ and to conclude that the set $\{u^*(\cdot, t), t \geqslant \delta > 0\}$ is relatively compact in $C^3(-\infty, \infty)$. A Lyapunov functional is then used to show that the limit set consists of just one travelling wave solution.

3. If the initial value ϕ is monotonic, then it is standard that u remains monotonic in x for all t. Hence, we can change to u, t as independent variables, with $v = u_x$ as the dependent variable. Differentiating (1) with respect to x, we obtain the corresponding problem for v:

$$v_t - v^2 v_{uu} + f v_u - f_u v = 0 \qquad (0 < u < 1, t > 0), \tag{2}$$

with

$$v(0, t) = 0, \quad v(1, t) = 0, \quad v(u, 0) = \Phi(u). \tag{3}$$

A travelling wave solution of (1) is a steady solution of (2), and we are interested in solutions for which $u(x, t)$ is monotonic decreasing in x, so that $v(u, t) < 0$ for u in (0, 1).

THEOREM C. *If* $f \in C^1[0, 1]$, *with* $f(0) = f(1) = 0$, *and if* $f(u) \leqslant 0$ *for u sufficiently small, while* $f(u) \geqslant 0$ *for u sufficiently near* 1, *then there is at most one steady solution of* (2) *satisfying* $v(0) = v(1) = 0$, $v < 0$ *in* (0, 1).

The steady form of (2) integrates to give $v_u + f/v = \text{constant} = -c$, say, c being in fact the wave-speed. Theorem C is proved by showing that there is a monotonic dependence of v on c, and this monotonicity is also used to discuss existence of steady solutions. If f has just one interior zero in (0, 1), then there does exist a (unique) steady negative solution (with zero boundary data) over [0, 1], and there is associated with this a characteristic wave-speed. If f has more interior zeros, the situation is more complicated.

THEOREM D. *Suppose that* [0, 1] *is divided into p subintervals* $[u_0, u_2]$, $[u_2, u_4], \ldots, [u_{2p-2}, u_{2p}]$, *where* $u_0 = 0$, $u_{2p} = 1$, *and that in each subinterval* (u_{2r}, u_{2r+2}) *there exists a point* u_{2r+1} *such that either*

$$f \leqslant 0 \quad in\ (u_{2r}, u_{2r+1}), \quad f > 0 \quad in\ (u_{2r+1}, u_{2r+2}), \quad \int_{u_{2r}}^{u_{2r+2}} f\,du > 0,$$

or

$$f < 0 \quad in\ (u_{2r}, u_{2r+1}), \quad f \geqslant 0 \quad in\ (u_{2r+1}, u_{2r+2}), \ \int_{u_{2r}}^{u_{2r+2}} f\, du < 0,$$

or

$$f < 0 \quad in\ (u_{2r}, u_{2r+1}), \quad f > 0 \quad in\ (u_{2r+1}, u_{2r+2}).$$

Then there exists a subset of $\{u_{2r}\}$, say $\{U_i\}$, $i = 0, \ldots, k$, *with* $U_0 = 0$, $U_k = 1$, *such that there is a (unique) steady negative solution of* (2) *(with zero boundary data) over* $[U_i, U_{i+1}]$, *but not over any* $[u_{2r}, u_{2s}]$ *unless it is a subinterval of some* $[U_i, U_{i+1}]$. *Further, if* c_i *is the wave-speed associated with* $[U_i, U_{i+1}]$, *then* $c_i \geqslant c_{i+1}$.

The physical interpretation of this is that the travelling waves corresponding to the subintervals $[u_{2r}, u_{2r+2}]$ of any $[U_i, U_{i+1}]$ have merged into a single travelling wave, but the travelling wave over $[U_i, U_{i+1}]$ is faster than that over $[U_{i+1}, U_{i+2}]$, since $c_i \geqslant c_{i+1}$, so that the two are moving apart (or at least not closing) and no single travelling wave can embrace them both.

By applying the maximum principle and ideas of sub- and super-solutions to the problem (2)–(3), we obtain

THEOREM E. *If f satisfies the conditions of Theorem* D, *and* $\Phi < 0$ *in* (0,1), *then the solution of* (2)–(3) *converges uniformly in each* $[U_i, U_{i+1}]$ *as* $t \longrightarrow \infty$ *to the steady negative solution (with zero boundary data) over* $[U_i, U_{i+1}]$.

This theorem can be interpreted with x and t as independent variables and leads to a result comparable with Theorem A.

REFERENCES

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