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ON THE CLASSIFICATION OF TAUT SUBMANIFOLDS

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All terminology will be smooth. A submanifold $K^{2n} \overset{i}{\hookrightarrow} M^{2n+2}$ is *taut* if $\pi_i(U, \partial U) = 0$ for $i \leqslant n$, where $U = (M\text{-neighborhood } K)$. Examples are: nonsingular algebraic hypersurfaces in CP^n (this follows from the Lefschetz theorem on hyperplane sections), simple knots (see [L]), the spines (see [M]). Every codimension-2 homology class contains taut representatives (see [K-M]), and the set of taut submanifolds is closed under connected sum (of pairs) with $(S^n \times S^n \overset{\text{standard}}{\hookrightarrow} S^{2n+2})$. Taut submanifolds are "almost canonical" in the sense of [Q], and from this viewpoint it is readily seen that if $n \geqslant 3$, every $K^{2n} \overset{i}{\hookrightarrow} M^{2n+2}$ with i n-connected is concordant to $K^{2n} \overset{i'}{\hookrightarrow} M^{2n+2}$ taut.

If M^{2n+2} is simply connected, the homology groups of K^{2n}, taut, are completely determined by the homology of M^{2n+2} except for $B_n(K^{2n})$. A lower bound on $B_n(K)$ in terms of $i_*[K^{2n}]$ and the cohomology ring of M^{2n+2} has been obtained in [T-W]. In [F1] we have proven Theorem 1, which provides a partial converse to Theorem 2.2 of [T-W] for $M \cong CP^{n+1}$, $n > 2$ odd, and $i_*[K] =$ a prime, p, multiple of the generator of $H_{2n}(CP^{n+1}; Z)$. Interestingly, if $p > 3$, the nonsingular algebraic hypersurfaces V are not the simplest taut submanifolds in their homology class, but may be decomposed as $V = K \#_{l\text{-copies}} S^n \times S^n$, $l > 0$, for some taut submanifold K.

We do not know if this is true for $n = 1$. If it were, there would be surfaces imbedded in CP^2 with genus smaller than that of the nonsingular algebraic hypersurfaces to which they are homologous. This would contradict Thom's conjecture.

Statement of Theorem 1. *Let M^{2n+2} be a simply-connected, oriented, smooth $(2n + 2)$-manifold, n odd > 1. Let $x \in H^2(M^{2n+2}; Z)$ generate a free summand of $H^2(M^{2n+2}; Z)$. Let p be any prime. Set*

$$\bar{s}_{\text{even}} = \max\{4, (\cosh(p - 2k)x)(\operatorname{sech}(px))(L(M))[M] \mid 0 < k < p\},$$

$$\bar{s}_{\text{odd}} = \max\{3, (\cosh(p - 2k)x)(\operatorname{sech}(px))(L(M))[M] \mid 0 < k < p\},$$

where L is the Hirzebruch polynomial.

For all integers $h \geqslant 0$, there exists a taut submanifold $K_h \overset{i}{\hookrightarrow} M$ with

$$M \cap px = i_*[K_h], \tag{1}$$

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and

$$
(2)\qquad
\begin{aligned}
B_n(K_h) &= \overline{s}_{\text{even}} + 6T_n(M) - 2B_n(M) + B_{n+1}(M) + 2h, \\
&\qquad\qquad\qquad\qquad \text{if } B_{n+1}(M) \text{ is even.} \\
&= \overline{s}_{\text{odd}} + 6T_n(M) - 2B_n(M) + B_{n+1}(M) + 2h, \\
&\qquad\qquad\qquad\qquad \text{if } B_{n+1}(M) \text{ is odd,}
\end{aligned}
$$

$B_n(M) = \operatorname{rank} H_n(M; Z)/\text{Torsion}$, $T_n(M) = \operatorname{rank} H_n(M) = \operatorname{rank} H_n(M; Z)$.

We now state two theorems, proved in [F2], which indicate to what extent the diffeomorphism class of a taut submanifold is fixed by $B_n(K)$.

THEOREM 2. *If M^{2n+2} is a compact, simply connected, smooth $(2n + 2)$-manifold, n odd $\geqslant 3$, and $K_0^{2n} \xhookrightarrow{i_0} M^{2n+2}$ and $K_1^{2n} \xhookrightarrow{i_1} M^{2n+2}$ are n-connected inclusions of closed submanifolds with $(i_0)_*[K_0] = (i_1)_*[K_1] \in H_{2n}(M^{2n+2}; Z)$, then if $B_n(K_0) = B_n(K_1)$, K_0 is diffeomorphic to K_1.*

THEOREM 3. *Assume M^{2n+2} is a simply-connected smooth $(2n + 2)$-manifold, n even, $\geqslant 2$, with $H_n(M; Z) = 0$. If i_0 and i_1 are as above, then if the intersection pairings on $H_n(K_0; Z)$/Torsion and $H_n(K_1; Z)$/Torsion are isometric, K_0 is diffeomorphic to K_1.*

If M^{2n+2}, n odd, $\geqslant 3$, is simply-connected, it follows from Theorem 2 that there is a simplest taut submanifold representing $i_*[K]$, K_0, and every other is of the form $K_l = K_0 \#_{l\text{-copies}} S^n \times S^n$. This, together with a previous remark, yields a complete classification of taut submanifolds in a homotopy CP^{n+1}, n odd, $\geqslant 1$, representing a prime multiple of the generator of $H_{2n}(CP^{n+1}; Z)$.

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