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ABSOLUTE CONTINUITY AND WEAK COMPACTNESS

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In this paper we discuss a class of *a priori* inequalities of a type appearing frequently in linear partial differential equations and outline the connection with the class of all weakly compact operators. The point of departure is the well-known result due to Bartle, Dunford and Schwartz [1] asserting the existence of a control measure for each weakly compact operator given on a space $C(S)$ and the reader can remark easily that our main result (Theorem 4) provides a more precise estimate than that obtained in [1]. Estimates of the same type hold also for p-absolutely summing operators ($1 \leqslant p < \infty$) as well as for the compact ones, which shows an interesting connection between these classes. Finally we extend the well-known criterion of compactness in a space l_p ($1 \leqslant p < \infty$) by proving that, on a space which does not contain an isomorph of l_1, the compact operators are completely determined by certain *a priori* inequalities.

The details will appear elsewhere.

Let X, Y be two Banach spaces and let p be a continuous seminorm on X. The following definition extends considerably a basic concept in the theory of vector measures:

1. DEFINITION. An operator $T \in L(X, Y)$ is said to be *absolutely continuous* with respect to p (i.e., $T \ll p$) if the following equivalent conditions hold:

(AC_1) for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if $\|x\| \leqslant 1$, $p(x) < \delta$, then $\|Tx\| < \epsilon$;

(AC_2) for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $\|Tx\| \leqslant \epsilon\|x\| + \delta p(x)$ whenever $x \in X$;

(AC_3) given a bounded sequence $\{x_n\}_n \subset X$, then either there exists a positive constant $c > 0$ such that $\|T(x_n)\| \leqslant cp(x_n)$ for all $n \in \mathbf{N}$ or there exists a subsequence $\{x_{n_k}\}_k$ such that $T(x_{n_k}) \longrightarrow 0$.

This notion is implicit in many papers on partial differential equations in which case p is associated with an inner product. Inequalities such as Gårding's or Friedrichs' are consequences of the fact that suitable operators are absolutely continuous.

A local condition for absolute continuity was introduced in [5] for operators given on Banach lattices and used to describe the structure of Banach lattices

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having an order continuous topology. (See also [6].)

In the following, by an absolutely continuous operator (abbreviated *a.c. operator*) we shall always mean an operator which is a.c. with respect to a prenuclear seminorm p [i.e., a seminorm p such that

$$p(x) \leqslant \int |\langle x, x^*\rangle| \, d\mu(x^*),$$

where μ is a positive Radon measure on the unit ball of X^*].

The results below yield that each r-absolutely summing operator ($1 \leqslant r < \infty$) and each compact operator is a.c. Using the Riesz convexity theorem one can obtain the same conclusion for inclusion mappings $i_q\colon l_1 \longrightarrow l_q$, $1 < q \leqslant \infty$ (communicated to the author by D. Lewis).

2. REMARK. An operator whose restriction to an infinite dimensional subspace is an isomorphism into cannot be a.c. (Use Dvoretzky's-Rogers' result in [2] and (AC_2).) In particular, if $i \in L(X, Y)$ is an isomorphism into and $T \in L(X, Y)$ is a.c., then $\operatorname{Ker}(i + T)$ is finite dimensional.

The product of an a.c. operator and a continuous mapping is always a.c., and thus the a.c. operators constitute a Banach ideal of operators in the sense of Pietsch. An a.c. operator maps bounded sequences into weak Cauchy sequences (use the main result in [7]) and Lebesgue's theorem on dominated convergence yields that each a.c. operator maps weak Cauchy sequences into convergent sequences. Therefore the product of two a.c. operators is a compact one.

3. REMARK. An operator T is a.c. iff T^{**} is a.c. Combining with the results above we check that every a.c. operator on an L_∞-space (i.e., a Banach space whose topological dual is isomorphic to a complemented subspace of some $L_1(\mu)$) is weakly compact. The converse is a consequence of the following:

4. THEOREM. *Let X be an L_∞-space and let A be a bounded subset of X^*. The following assertions are equivalent*:

(a) *A is $\sigma(X^*, X^{**})$ relatively compact*;

(b) *there is a prenuclear seminorm p on X such that $A \ll p$ uniformly* [*i.e., for every $\epsilon > 0$ we can find a $\delta = \delta(\epsilon) > 0$ such that $\|x\| \leqslant 1$, $p(x) < \delta$ implies that* $\sup\{|\langle x, x^*\rangle|, x^* \in A\} < \delta$].

5. COROLLARY. *An operator given on an L_∞-space is weakly compact iff it is a.c.*

6. COROLLARY. *Each compact operator is a.c.*

Use Corollary 4 in [4, p. 24].

7. COROLLARY. *Let X be a Banach space which does not contain an isomorph of l_1. Then a bounded subset $A \subset X^*$ is relatively compact iff there exists a prenuclear seminorm p on X such that $A \ll p$ uniformly.*

The assumption about l_1 is essential. In fact, if X contains l_1 then there exists an integral operator $T \in L(X, L_1(0, 1))$ which is not compact.

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