BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 5, September 1975

A STRUCTURE SHEAF FOR A NONCOMMUTATIVE NOETHERIAN RING

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Communicated by Barbara L. Osofsky, June 2, 1975

Throughout we assume that R is a left noetherian ring, not necessarily commutative. R-modules are left modules, $[M]$ denotes the isomorphism class and $E(M)$ the injective hull of a module M.

The (left) *spectrum* of R, denoted Spec R, is taken to be the set of isomorphism classes of indecomposable injective modules. We denote by $G(R)$ the directed graph whose set of vertices is the set Spec R and such that if $[V_1]$, $[V_2] \in \operatorname{Spec} R$ there is a directed edge from $[V_1]$ to $[V_2]$ in $G(R)$ if and only if there exist critical (see [2, p. 9]) submodules S_1 and S_2 of V_1 and V_2 and a short exact sequence $0 \longrightarrow S_1 \longrightarrow A \longrightarrow S_2 \longrightarrow 0, A \subseteq V_1$. For a left ideal I of R define $K(I)$ to be the subset of Spec R consisting of $[V]$ such that there exists $[W] \in \operatorname{Spec} R$, a nonzero map $\alpha: R/I \longrightarrow W$, and a directed path in $G(R)$ from $[W]$ to $[V]$. If there is a nonzero module map from W to V there is a directed path in $G(R)$ from $[W]$ to $[V]$.

PROPOSITION 1. *The collection of subsets of* Spec R *of the form* $K(I)$, *I a left ideal of R, is a basis for the closed sets of a topology on R. If R is commutative then* Spec R *is homeomorphic to the classical spectrum with its Zariski, or hull-kernel, topology.*

For each $x \in \operatorname{Spec} R$ we fix an indecomposable injective module V_x such that $[V_x] = x$. For a subset Y of Spec R we define $V_Y = \coprod_{y \in Y} V_y$, $E_Y = \operatorname{End}_R(V_Y)$, and $R_Y = \operatorname{End}_{E_Y}(V_Y) = \operatorname{Biend}_R(V_Y)$. We regard V_Y as a right E_Y-module. We define a presheaf of rings over Spec R by letting, for each open subset U, the ring of sections over U be R_U, with the obvious restriction maps.

THEOREM 2. *The above presheaf is a sheaf and for each open subset U of* Spec *R, R_U is naturally identified with the quotient ring of R with respect to the torsion theory determined by the injective module V_U. In particular, $R_{\operatorname{Spec} R}$ is naturally identified with R.*

If R is commutative the sheaf constructed above reduces to the usual structure sheaf.

If M is an R-module the *support* of M, denoted Supp M, is the set of those $[V] \in \operatorname{Spec} R$ such that M is not V-torsion. (For definitions and basic properties on torsion theories, see [6].) When R is commutative, the above definition of the

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AMS (MOS) subject classifications (1970). Primary 16A08, 16A46, 16A52.

support of M agrees with the usual definition.

PROPOSITION 3. *If* Supp M *is the disjoint union of relatively open sets* X_1, X_2, *then* M *has a unique decomposition,* $M = M_1 \oplus M_2$, *where* $X_i =$ Supp M_i, $i = 1, 2$. *In particular, if* Spec R *is the disjoint union of open subsets* X_1, X_2, *then* $R = R_1 \oplus R_2$, *where each* R_i *is a two-sided ideal and* Spec R *is naturally homeomorphic to* X_i, $i = 1, 2$.

If M is an R-module we define a presheaf of modules over Spec R by letting for an open set U, the module of sections over U be M_U, where M_U is the quotient module of M with respect to the torsion theory determined by V_U, with the obvious restriction maps.

THEOREM 4. *If each open subset of* Spec R *is compact, the above presheaf is a sheaf and the construction yields a functor from the category of* R-*modules to the category of sheaves of modules over* Spec R.

If R is commutative the sheaf constructed above reduces to the usual sheaf.

A ring map $\phi: R \longrightarrow S$ between noetherian rings R and S is *special* if for each indecomposable injective S-module V, $E[{}_R V]$ is the direct sum of pairwise isomorphic indecomposable injective R-modules. If ϕ is special there is a naturally defined map ϕ^*: Spec $S \longrightarrow$ Spec R such that ϕ^*: $[{}_S V] \longrightarrow [{}_R W]$ where ${}_R W$ is an indecomposable direct summand of $E[{}_R V]$. We do not know under what conditions ϕ^* is continuous; however, ϕ^* does have a property that is closely related to continuity.

We say that a subset Y of Spec R is *closed under generization* if Spec $R \setminus Y$ has the property that it contains the closure of each of its points. So open sets are closed under generization.

THEOREM 5. *Assume that* $\phi: R \longrightarrow S$ *is special. If* $Y \subseteq$ Spec R *is closed under generization, then* $\phi^{*-1}(Y) \subseteq$ Spec S *is closed under generization.*

A ring map $\phi: R \longrightarrow S$ is an *extension* if S is generated as left R-module by the set $\{c \in S \mid c\phi(r) = \phi(r)c, \text{ all } r \in R\}$.

PROPOSITION 6. *If* $\phi: R \longrightarrow S$ *is an epimorphism in the category of rings, then* ϕ *is special. If* ϕ *is an extension, then* ϕ *is special.*

Several topologies on Spec R have been defined in [4], [1], and [5]. None of these agree with the topology defined above. For example, if R is the ring of $n \times n$ upper triangular matrices over a field, then Spec R has n points. In each of the topologies in the listed references, Spec R is a discrete space, while our topology is the order topology on a linearly ordered set. Our topology is very like that given in [3] where, as in the other references, the noetherian condition is not assumed. We do not know if the topology defined in [3] agrees with our topology for arbitrary left noetherian rings.

An expanded version of this paper, with proofs and examples, will be published elsewhere.

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