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NORMAL SELF-INTERSECTIONS OF THE CHARACTERISTIC VARIETY

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Let $P = P_1P_2 + Q$ be a linear partial differential operator on $\mathbf{R}^N$ with P_1 and P_2, of orders m_1 and m_2, respectively, strictly hyperbolic with respect to the first variable and Q of order $m_1 + m_2 - 2$. Although the characteristic variety of P may have self-intersections, the hyperbolicity of P_1 and P_2 implies local solvability for $Pu = f$; indeed the Cauchy problem for P is locally solvable. In this note we shall consider the propagation of singularities near the simplest type of point $z_0 \in T^*\mathbf{R}^N\backslash 0$ where the principal symbol $p = p_1p_2$ of P has a multiple zero.

We shall suppose that the characteristic varieties $A(P_1)$ and $A(P_2)$ of P_1 and P_2 intersect normally at z_0, that is, $dp_1(z_0)$ and $dp_2(z_0)$ are linear independent. In addition, it will be assumed that the Poisson bracket $\{p_1, p_2\}(z_0) \neq 0$. This latter assumption means that the Hamiltonian vector fields H_{p_1} and H_{p_2} are not tangent to $A(P_1) \cap A(P_2)$ at z_0. So, the two forward pointing bicharacteristics (of p_1 and p_2) through z_0 consist, near z_0, of nonsingular points of $A(P)$, except for z_0 itself. Let these curves be denoted by c_i; $\mathbf{R} \supset I \ni \rho \longrightarrow c_i(\rho)$ where I is an open interval containing 0, $c_i(0) = z_0$ and $(c_i)_*(d/d\rho) = H_{p_i}$. It will be assumed that I is chosen so small that

$$(1) \qquad c_i(I) \cap A(P_j) = \{z_0\}, \qquad i \neq j.$$

If I^+ (I^-) is the open interval consisting of the positive (negative) points in I then, by Hörmander's Theorem [4, Theorem 3.2.1], if $u \in \mathcal{D}'(\mathbf{R}^N)$, $z_0 \notin WF(Pu)$ and I is chosen so small that

$$(2) \qquad c_i(I) \cap WF(Pu) = \emptyset, \qquad i = 1, 2,$$

then either $c_i(I^\pm) \subset WF(u)$ or $c_i(I^\pm) \cap WF(u) = \emptyset$ separately for the four choices of sign and bicharacteristic. Hörmander's Theorem does not, however, give any information as to whether $z_0 \in WF(u)$ or not.

THEOREM. *Suppose $A(P_1)$ and $A(P_2)$ intersect normally at z_0 and that $\{p_1, p_2\}(z_0) \neq 0$. If $u \in \mathcal{D}'(\mathbf{R}^N)$, $z_0 \notin WF(Pu)$ and I is chosen so small that* (1) *and* (2) *hold, then either $c_i(I^+) \cap WF(u) = \emptyset$ for $i = 1, 2$, or $c_i(I^-) \cap WF(u) = \emptyset$ for $i = 1, 2$ implies $z_0 \notin WF(u)$ and $c_i(I) \cap WF(u) = \emptyset$ for $i = 1, 2$.*

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The main part of the proof consists in the construction of suitable microlocalizing pseudodifferential operators and this is carried out by a modification of the method used in Nirenberg's paper [5].

Similar results hold for first order symmetric hyperbolic systems and so lead to generalized Poisson relations for the spectral measure of the associated elliptic operator (compare Chazarain [1], Duistermaat and Guillemin [2]). Let A be an $m \times m$ first order system of classical elliptic pseudodifferential operators defined, for simplicity, on the m-fold direct sum of the half-density bundle over a compact manifold M and suppose that the principal symbol a of A is symmetric and uniformly diagonalizable (i.e. has eigenvalues and eigenvectors smoothly defined on $T^*M\backslash 0$). Suppose further that at each point $z \in T^*M\backslash 0$ either a has m distinct eigenvalues or $m - 1$ distinct eigenvalues and the coincident eigenvalues $\lambda_i(z) = \lambda_j(z)$, $i \neq j$, satisfy $d\lambda_i(z) \neq d\lambda_j(z)$ and $\{\lambda_i, \lambda_j\}(z) \neq 0$. Then the spectral density of A,

$$\sigma(\mu) = \sum_k \delta(\mu - \mu_k),$$

where $\{\mu_k\}$ are the eigenvalues of A, satisfies the following

THEOREM. *The singularities of the Fourier transform $\hat{\sigma}(t)$ of σ occur only at points $|t| = T$ such that there exists a piecewise smooth closed curve of parameter length T each of whose smooth segments is an integral curve of one of the Hamiltonian fields H_{λ_j}.*

The proofs of these and other related results will appear elsewhere. It should be noted that the proof is not constructive and does not, for example, produce a microlocal parametrix for P or $\partial_t - A$ from which the singularities can be computed directly (see, however, Guillemin [3]).

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