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CONCAVITY OF MAGNETIZATION FOR A CLASS OF EVEN FERROMAGNETS[1]

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1. **Introduction.** Let E be the set of even probability measures which satisfy $\int \exp(kx^2)\rho(dx) < \infty$ for all $k \geq 0$ sufficiently small. Given an integer $N \geq 1$, real numbers $h \geq 0$ and $J_{ij} \geq 0$, $1 \leq i \leq j \leq N$, and measures $\rho_i \in E$, $1 \leq i \leq N$, we define [11, p. 273] real-valued random variables X_i, $1 \leq i \leq N$, with the joint distribution

$$\tau_h(dx_1, \ldots, dx_N) \tag{1}$$
$$= \frac{\exp(\Sigma_{1\leq i<j\leq N} J_{ij}x_ix_j + h\Sigma_{1\leq i\leq N} x_i)\rho_1(dx_1) \ldots \rho_N(dx_N)}{Z(h)}.$$

$Z(h)$, the partition function, is given by the formula

$$Z(h) = \int_{R^N} \cdots \int \exp\left(\sum J_{ij}x_ix_j + h\sum x_i\right)\rho_1(dx_1) \cdots \rho_N(dx_N). \tag{2}$$

The J_{ij} are assumed to be so small that the integral in (2) converges for all $h \geq 0$. The inequalities we discuss are to hold for all $h \geq 0$ and all $J_{ij} \geq 0$ subject only to this restriction. The choice of ρ_i as the Bernoulli measure $b(dx) = \frac{1}{2}(\delta(x-1) + \delta(x+1))$ gives the classical Ising model.

We define the average magnetization per site, $m(h)$, by the formula

$$m(h) = \frac{1}{N}\frac{d}{dh} \ln Z(h) = \frac{1}{N}\sum_{i=1}^{N} E\{X_i\} \tag{3}$$

and consider inequalities on $m(h)$ and its derivatives. While the inequalities $m(h) \geq 0$, $dm(h)/dh \geq 0$ hold for any $\rho_i \in E$ [7, pp. 76–77], the concavity of $m(h)$, i.e.

$$d^2m(h)/dh^2 \leq 0, \tag{4}$$

requires that further restrictions be placed on the ρ_i. Essentially, (4) is known to hold only in the Ising case and in models which can be built out of Ising models in a suitable way [4], [6]. Measures for which (4) fails are known [6].

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The usual approach to (4) is first to prove the stronger (GHS) inequalities [5]

$$(5)\qquad \frac{\partial^3}{\partial h_i \partial h_j \partial h_k} \ln Z(h_1, \ldots, h_N) \leqslant 0, \quad \text{all } 1 \leqslant i, j, k \leqslant N,\ h_i \geqslant 0,$$

where

$$Z(h_1, \ldots, h_N) = \int_{R^N} \cdots \int \exp\left(\sum J_{ij} x_i x_j + \sum h_i x_i\right) \rho_1(dx_1) \cdots \rho_N(dx_N).$$

Instead, we shall prove (4) directly for many new measures using a technique which reduces consideration to the case $N = 1$. Afterwards, we shall return to (5).

We state two implications of these inequalities. The first shows that the requirement that the ρ_i in (1) have Gaussian falloff is only an apparent restriction.

THEOREM 1. *Let ρ be an even probability measure satisfying $\int \exp(kx)\rho(dx) < \infty$ for all $k \geqslant 0$. Assume that* (4) *holds for $N = 1$* (set $\rho_1 = \rho$). *Then ρ is in $\mathcal{E}$.*

The next theorem (known for fourth degree polynomial V [3], [10]) on the spectrum of certain differential operators is a striking consequence of (5).

THEOREM 2. *Let $V(x)$ be an entire function with the expansion*

$$(6)\quad V(x) = \sum_{k=1}^{\infty} a_k x^{2k}, \quad a_k \geqslant 0 \text{ for } k \geqslant 2, \quad a_1 \text{ real } (a_1 > 0 \text{ if all } a_k = 0).$$

Let E_1, E_2, E_3, be the three smallest eigenvalues of the differential operator $-\frac{1}{2} d^2/dx^2 + V(x)$ on $L^2(R^1; dx)$. Then $E_3 - E_2 \geqslant E_2 - E_1$.

By Theorems 4 and 5 below, we shall see that (5) is satisfied for the measures

$$(7)\qquad \rho_i(dx) = c \exp(-V(x))\, dx, \quad c \text{ a normalization constant},$$

if V is as in (6). This is the main ingredient needed to prove Theorem 2 [10].

2. **The class $\mathcal{G}_-$.** Below, we define a subset $\mathcal{G}_-$ of measures in $\mathcal{E}$ for which we have the following result.

THEOREM 3. *If $\rho_1, \ldots, \rho_N \in \mathcal{G}_-$, then* (4) *holds.*

For the proof, we use a closure property of $\mathcal{G}_-$ in order to reduce to the case $N = 1$. We call this property the closure of $\mathcal{G}_-$ under *ferromagnetic unions.*

(C) *Let $Y_1, \ldots, Y_N$ be real-valued random variables with joint distribution τ_0 (see* (1)*). Let $\mathcal{F}_0$ be the class of all distributions of sums $\Sigma_{1 \leqslant i \leqslant N} r_i Y_i$*

for arbitrary choice of $N \geqslant 1, r_i \geqslant 0, J_{ij} \geqslant 0$, *and* $\rho_1, \ldots, \rho_N \in G_-$. *Then* $F_0 \subseteq G$.

The partition function $Z(H)$ in (2) can be written as

$$Z(h) = Z(0)E\left\{\exp\left(h \sum_{1\leqslant i\leqslant N} Y_i\right)\right\};$$

i.e., $m(h)$ is related to the average magnetization $\widetilde{m}(h)$ for a *single site system* (with spin $\Sigma_{1\leqslant i\leqslant N} Y_i$ at the single site) by the formula $m(h) = N^{-1}\widetilde{m}(h)$. Hence, Theorem 3 for general N is a consequence of (C) once we have proved Theorem 3 for $N = 1$. We do the latter in §3.

The next theorem indicates which measures belong to G_-.

THEOREM 4. G_- *contains the Bernoulli measure* $b(dx)$ *and all measures of the form* (7), *where* $V(x)$ *is as in* (6). *Also,* G_- *contains the distributions of all weak limits of* $Y^{(N)} \in F_0$ *which satisfy* $\sup_N E\{(Y^{(N)})^2\} < \infty$.

The first part of Theorem 4 will be proved after G_- is defined.

DEFINITION. *Given* $\rho \in E$, *let* $W_1, \ldots, W_4$ *be four independent copies of a random variable distributed by* ρ. *The vector* $\vec{m} = (m_1, \ldots, m_4)$, *where each* m_i *is a nonnegative integer, is said to be odd if each* m_i *is odd. Let* $W = (W_1, \ldots, W_4)$; *take* A *to be the orthogonal matrix*

$$2^{-1}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

and define

$$(A\vec{W})_i = \sum_{j=1}^{4} A_{ij} W_j, \quad (A\vec{W})^{\vec{m}} = (A\vec{W})_1^{m_1} \cdots (A\vec{W})_4^{m_4}, \quad \mu_\rho(\vec{m}) = E\{(A\vec{W})^{\vec{m}}\}.$$

We define

$$G_- = \{\rho \colon \rho \in E \text{ and } \mu_\rho(\vec{m}) \leqslant 0 \text{ for all } \vec{m} \text{ odd}\}.$$

The condition that $\mu_\rho(\vec{m}) \leqslant 0$ for all $\vec{m}$ odd implies an infinite string of inequalities satisfied by the moments of a measure $\rho \in G_-$. We refer the reader to [8] and [9], where other moment inequalities are derived for measures which satisfy the Lee-Yand theorem.

To show that $b(dx) \in G_-$, consider

$$\mu_b(\vec{m}) = \frac{1}{16} \sum_{x_i=\pm 1} (x_1 + x_2 + x_3 + x_4)^{m_1}(-x_1 + x_2 - x_3 + x_4)^{m_2} \cdot (-x_1 - x_2 + x_3 + x_4)^{m_3}(x_1 - x_2 - x_3 + x_4)^{m_4}.$$

Each of the 16 summands is either negative or zero according to whether an odd number or an even number of the x_i equal $+1$.

Given a measure ρ as in (7), the joint distribution of the random vector

$A\vec{W}$ has the form

$$\exp(-f(z_1, \ldots, z_4))\exp(g(z_1, \ldots, z_4))\,dz_1 \cdots dz_4,$$

where f is an odd function of each z_i and $f \geqslant 0$ when each $z_i \geqslant 0$ and g is an even function of each z_i. Greater weight is thus given to those values of $z_1, \ldots$ z_4 where an odd number of the z_i are negative than where an even number of the z_i are negative. From this, it can be shown that $\rho \in \mathcal{G}_-$.

We also have a characterization of Gaussian measures in terms of $\mathcal{G}_-$.

THEOREM 5. *Given $\rho \in \mathcal{E}$, the numbers $\mu_\rho(\vec{m}) = 0$ for all $\vec{m}$ odd if and only if ρ is an even Gaussian measure.*

Inequality (5) holds under the same hypothesis as (4).

THEOREM 6. *If $\rho_1, \ldots, \rho_N \in \mathcal{G}_-$, then* (5) *holds.*

The proof makes use of multivariate versions of the $\mathcal{G}_-$ inequalities. Let $Y_i^{(j)}$, $1 \leqslant j \leqslant 4$, be independent copies of Y_i (see (C)) and define $\vec{Y}_i = (Y_i^{(1)}, \ldots, Y_i^{(4)})$. Then

$$E\{(A\vec{Y}_1)^{\vec{m}_1} \ldots (A\vec{Y}_N)^{\vec{m}_N}\} \leqslant 0$$

whenever $\rho_1, \ldots, \rho_N \in \mathcal{G}_-$ and $\vec{m}_1 + \ldots + \vec{m}_N$ is odd.

3. **Proof of Theorem 3 for $N = 1$.** Given $\rho \in \mathcal{G}_-$, we write $Z(h) = \int \exp(hx)\rho(dx)$, $h \geqslant 0$. We have ($'$ denotes d/dh)

$$\begin{aligned}(\ln Z)''' &= (Z^2Z''' - 3ZZ'Z'' + 2(Z')^3)/Z^3 \\ &= \frac{1}{Z^4}\int_{R^4}\!\!\cdots\int \left[\frac{\partial^3}{\partial h_1^3} - 3\frac{\partial^3}{\partial h_1^2 \partial h_2} + 2\frac{\partial^3}{\partial h_1 \partial h_2 \partial h_3}\right] \\ &\qquad \cdot e^{\langle \vec{h}, \vec{x}\rangle}\rho(dx_1)\ldots\rho(dx_4)|_{h_i = h},\end{aligned}$$

where $\vec{h} = (h_1, \ldots, h_4)$, $\vec{x} = (x_1, \ldots, x_4)$, and $\langle\cdot,\cdot\rangle$ is the R^4 inner product. Define $\vec{s} = (s_1, \ldots, s_4) = \vec{h}A^t$. An easy calculation [1, Appendix] shows that the last integral equals

$$\frac{2}{Z^4}\int_{R^4}\!\!\cdots\int \frac{\partial^3}{\partial s_2 \partial s_3 \partial s_4} e^{\langle \vec{s}, A\vec{x}\rangle}\rho(dx_1)\cdots\rho(dx_4)|_{h_i = h}.$$

Expanding the exponential and carrying out the integration, we find

$$\begin{aligned}(\ln Z)''' = \frac{2}{Z^4}\sum_{n=0}^{\infty}\ \sum_{m_1+\ldots+m_4=n} &\frac{m_2m_3m_4}{m_1!\cdots m_4!} \\ &\cdot \mu_\rho(\vec{m})s_1^{m_1}s_2^{m_2-1}s_3^{m_3-1}s_4^{m_4-1}|_{h_i=h}.\end{aligned}$$

But when each $h_i = h$, then $s_1 = 2h$, $s_2 = s_3 = s_4 = 0$. Also, $\mu_\rho((k, 1, 1, 1))$ can be shown to be zero unless k is odd. Hence

$$(8) \qquad (\ln Z)''' = \frac{2}{Z^4} \sum_{k\,\mathrm{odd};k \geqslant 0} \frac{(2h)^k}{k!} \mu_\rho((k, 1, 1, 1)),$$

which is negative since $\rho \in G_-$ and $h \geqslant 0$. This completes the proof.

In this proof, we did not need the full force of the assumption that $\rho \in G_-$; viz., that $\mu_\rho(\vec{m}) \leqslant 0$ for *all* $\vec{m}$ odd. However, the latter is needed to prove Theorem 6. Also, the set of measures $\vec{\rho}$ for which $\mu_\rho((k, 1, 1, 1)) \leqslant 0$ for all k odd is not necessarily closed under ferromagnetic unions.

REMARK. Proofs and related matter will appear in [2].

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