BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 5, September 1975

THE WALL OBSTRUCTION IN SHAPE AND PRO-HOMOTOPY, WITH APPLICATIONS

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Communicated by W. Wistar Comfort, January 23, 1975

1. **Geometrical results.** There exist (CW) complexes X homotopy dominated by finite complexes but not homotopy equivalent to finite complexes [8]. It is unknown if there are compact metric spaces (compacta) with this property. However, one may construct a compactum, Z, shape dominated by a finite complex but not shape equivalent (hence not homotopy equivalent) to a finite complex by the following trick. Let

$$X \underset{d}{\overset{u}{\rightleftarrows}} K$$

be a homotopy domination of X (above) by a finite complex K; $d \circ u$ is homotopic to 1. Form the inverse sequence

$$K \xleftarrow{u \circ d} K \xleftarrow{u \circ d} K \xleftarrow{u \circ d} \cdots .$$

This sequence is isomorphic to X in pro-homotopy [1]. Hence its inverse limit, Z, is a compactum shape equivalent to X: see [3]. Since homotopy theory and shape theory agree on complexes, Z has the required properties. By [8], K may be chosen two dimensional. So we will assume Z is two dimensional (and connected).

Embed Z in S^5 with $S^5 \backslash Z$ 1-ULC. Then $S^5 \backslash Z$ is not homeomorphic to the interior of a compact manifold: otherwise Z would be a shape deformation retract of a compact topological manifold neighborhood of Z in S^5, and such a neighborhood would have finite homotopy type. But, by Siebenmann's theory of I-regular neighborhoods [7], the end of $S^5 \backslash Z$ is *tame*, in the sense of [6]. So $S^5 \backslash Z$ has nonvanishing Siebenmann obstruction (a *strange* end [6]). So has $S^n \backslash Z, n > 5$.

The map $u \circ d: K \longrightarrow K$ is a homotopy idempotent, but it is not homotopic to a strict idempotent, *not even stably.* For, the inverse limit of the sequence obtained by repeating a strict idempotent is a (compact) ANR, and this compact ANR would be shape equivalent to Z, contradicting [9]. Details appear in [3].

2. **Shape and pro-homotopy.**

THEOREM 1. *Let Z be a connected compactum, $z \in Z$. The following are equivalent*: (i) Z *is shape dominated by a finite complex*; (ii) Z *is shape*

AMS (MOS) subject classifications (1970). Primary 55B99, 55D10; Secondary 57A99.

[1]Supported in part by NSF Grant PO 38761.

equivalent to a complex; (iii) *Z is shape dominated by a complex. Furthermore given* (i), (ii) *or* (iii), *there is an intrinisic "Wall obstruction"* $w(Z, z) \in \widetilde{K}^0(\check{\pi}_1(Z, z))$ *which vanishes if and only if Z is shape equivalent to a finite complex. All possible obstructions occur among two-dimensional compacta.*

($\check{\pi}_1(Z, z)$ is the Čech fundamental group; $\widetilde{K}^0(G)$ is the projective class group of the group G; pointed shape theory is understood here.)

The proof of (ii) implies (iii) uses an exact sequence [2, IX 3.2]. In the spirit of [2], we define the *homotopy limit* of a compactum Z and prove that this "large" complex is homotopy equivalent to the I-regular neighborhood of Z [7] whenever the latter exists. Details appear in [3].

3. Splitting homotopy idempotents.

THEOREM 2. *Let X be a complex and* $f\colon X \longrightarrow X$ *a map with* $f \circ f$ *pointedly homotopic to* $(\sim) f$. *There exist a complex P and maps*

$$P \underset{d}{\overset{u}{\rightleftarrows}} X$$

with $d \circ u \sim 1$ *and* $u \circ d \sim f$. *If X is finite dimensional so is P. If X is finite, P may be chosen finite if and only if a "Wall obstruction"* $w(f) \in \widetilde{K}^0(\pi_1(X))$ *vanishes. All possible obstructions are realized.*

The same proof gives a similar theorem in stable homotopy (compare [5]). The obstruction is trivial in that case. Details appear in [4].

ADDED IN PROOF. We have learned that other proofs of Theorem 2 are known to P. Freyd (using Brown's theorem) and to W. Holsztynski (using a homotopy direct limit).

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