BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 5, September 1975

AN EXTENSION OF KHINTCHINE'S INEQUALITY[1]

BY C. M. NEWMAN

Communicated March 20, 1975

Khintchine's inequality [4] states that if $\{X_j: j = 1, \ldots, N\}$ are independent identically distributed Bernoulli random variables ($X_j = \pm 1$ with equal probabilities), then for any choice of real a_j, and any $m = 2, 3, \ldots, X = \Sigma_j a_j X_j$ satisfies

$$E(X^{2m}) \leqslant ((2m)!/2^m m!)(E(X^2))^m. \tag{1}$$

This inequality implies [9, Chapter 5] that for $0 < p < \infty$, there exist positive constants A_p and B_p depending only on p (with $B_{2m} = ((2m)!/2^m m!)^{1/2m}$) such that

$$A_p\|X\|_2 \leqslant \|X\|_p \leqslant B_p\|X\|_2 \tag{2}$$

where $\|X\|_p$ denotes the p-norm, $(E(|X|^p))^{1/p}$. Khintchine's inequality in this form has many applications in which the $\{X_j\}$ are generally represented as Rademacher functions [9], [7], [3].

In this note we give an extension of Khintchine's inequality from the Bernoulli case to that of random variables of the following type:

DEFINITION. A random variable X is of *type* L if its moment generating function $E_X(z) \equiv E(\exp(zX))$ satisfies

(i) $\exists C \ni E_X(z) \leqslant \exp(Cz^2)$ for all real z and

(ii) $E_X(z) = 0 \Rightarrow z = i\alpha$ for some real α.

Symmetric random variables satisfying condition (i) have been called *subgaussian* by Kahane; they satisfy an inequality similar to but weaker than (1) [2, p. 87].

Theorem 1 below extends Khintchine's inequality to arbitrary linear combinations of independent random variables of type L while Theorem 2 treats the case of positive linear combinations of type L random variables with a particular kind of dependence (such as arises in models of ferromagnets). Complete proofs of these theorems together with further results concerning random variables of type L and applications of these results to statistical mechanics and quantum field theory will appear in [6].[2]

THEOREM 1. *If $\{X_j\}_{j=1}^N$ are independent (not necessarily identically distributed) random variables of type L, then the inequality* (1) *applies for any*

AMS (MOS) subject classifications (1970). Primary 42A36, 60G50; Secondary 42A56.

[1]Supported in part by NSF MPS 74-04870.

[2]Other related results are contained in a paper, *Gaussian correlation inequalities for ferromagnets*, by the author, which will appear in Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.

choice of real a_j *and any* $m = 2, 3, \ldots$ *to* $X = \Sigma_j a_j X_j$.

SKETCH OF PROOF. Since the X_j are independent, X is itself of type L. Hadamard factorization methods [1, Theorems 2.7.1 and 2.10.1] imply that for any random variable X of type L,

$$E_X(z) = \exp(bz^2)\prod_j (1 + (z/\alpha_j)^2), \tag{3}$$

for some $b \geq 0$ and $0 < \alpha_1 \leq \alpha_2 \leq \cdots$ with $\Sigma(1/\alpha_j)^2 < \infty$. We next note that $E_X(z) = \Sigma E(X^n)z^n/n!$ so that by (3),

$$E(X^2)/2 = b + \sum_j (1/\alpha_j)^2. \tag{4}$$

Now each Taylor coefficient of $(1 + (z/\alpha_j)^2)$ is bounded by the corresponding Taylor coefficient of $\exp((z/\alpha_j)^2)$ from which it follows by (3) and (4) that each Taylor coefficient of $E_X(z)$ is bounded by the corresponding one of $\exp(z^2 E(X^2)/2)$ which yields (1).

REMARK. X may satisfy (1) without being of type L as can be seen by considering the probability distribution

$$(1 - \beta)\delta(x) + \beta(\delta(x - 1) + \delta(x + 1))/2 \quad \text{for } 1/3 \leq \beta < 1/2.$$

THEOREM 2. *Suppose* $\{Y_j\}_{j=1}^N$ *are random variables whose joint probability distribution* ρ *on* $\mathbf{R}^N$ *is of the form*

$$\rho(y_1, \ldots, y_N) = C' \exp\left(\sum_{j,k=1}^N J_{jk} y_j y_k\right) \prod_{j=1}^N \mu_j(y_j), \tag{5}$$

with $J_{jk} \geq 0\ \forall j, k$, *and with each* μ_j *an even measure satisfying*:

(a) $\int \exp(by^2)d\mu_j(y) < \infty \quad \forall b > 0$, *and*

(b) $\int \exp(zy)d\mu_j(y) = 0 \Rightarrow z = i\alpha$ *for some real* α;

then for any choice of $\lambda_j \geq 0$, $X \equiv \Sigma \lambda_j Y_j$ *is of type* L *and thus satisfies* (1) $(m = 2, 3, \ldots)$.

SKETCH OF PROOF. Theorem 2 follows directly from the proof of Theorem 1 combined with a general version of the (Statistical Mechanics) Lee-Yang Theorem [5, Theorem 1.1].

Examples of measures μ satisfying (a) and (b)[3] (and thus examples of type L random variables) include:

$$\mu(y) = \sum_{k=0}^n \delta(y - (n - 2k)), \qquad n = 1, 2, \ldots \tag{6}$$

$$d\mu/dy = \begin{cases} 1, & |y| \leq A, \\ 0, & |y| > A, \end{cases} \quad A > 0, \tag{7}$$

[3]Many other examples can be found in various of Polya's papers on the location of zeros of entire functions.

$$(8) \qquad d\mu/dy = \begin{cases} (1-y^2)^{(d-2)/2}, & |y| \leqslant 1, \\ 0, & |y| > 1, \end{cases} \quad d > 0,$$

$$(9) \qquad d\mu/dy = \exp(-\lambda \cosh y), \qquad \lambda > 0,$$

$$(10) \qquad d\mu/dy = \exp(-ay^4 - by^2), \qquad a > 0 \quad [8].$$

When d is an integer, example (8) is the one-dimensional marginal distribution of the uniform distribution on the surface of the unit d-sphere (in $\mathbf{R}^{d+1}$).

REFERENCES

1. R. P. Boas, Jr., *Entire functions,* Academic Press, New York, 1954. MR **16**, 914.

2. J. P. Kahane, *Séries de Fourier aléatoires,* Deuxième édition multigraphiée, Les Presses de l'Université de Montréal, Montreal, 1967. MR **42** #3483.

3. T. Kawata, *Fourier analysis in probability theory,* Academic Press, New York, 1972.

4. A. Khintchine, *Über dyadische Brüche,* Math. Z. **18** (1923), 109–116.

5. C. M. Newman, *Zeros of the partition function for generalized Ising systems,* Comm. Pure Appl. Math. **27** (1974), 143–159.

6. ———, *Inequalities for Ising models and field theories which obey the Lee-Yang theorem,* Comm. Math. Phys. **41** (1975), 1–9.

7. A. Pełczyński, *A characterization of Hilbert-Schmidt operators,* Studia Math. **28** (1966/67), 355–360. MR **35** #7163.

8. B. Simon and R. B. Griffiths, *The* $(\phi^4)_2$ *field theory as a classical Ising model,* Comm. Math. Phys. **33** (1973), 145–164.

9. A. Zygmund, *Trigonometric series.* Vol. I, 2nd ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401