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CONVEXITY FOR A SIMPLY CONNECTED p-ADIC GROUP

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In [4] Kostant showed that the set of Iwasawa double cosets which intersect a given Cartan double coset in a semisimple Lie group corresponds to a certain convex subset in the Lie algebra of a maximal torus of the group. As a consequence, he established that representatives for the double cosets relative to a maximal compact subgroup of a semisimple Lie group may be chosen in the unipotent radical of a minimal parabolic subgroup. We announce here analogues of these results for a simply-connected p-adic group.

I wish to thank Roger Howe and Jacques Tits for valuable conversations during the time I was thinking about this problem.

Let $\mathbf{G}$ be a connected simply-connected semisimple algebraic group defined over a p-adic field Ω. Let G denote the group of Ω-rational points of $\mathbf{G}$. Then G is a locally compact totally disconnected group. Borel and Tits [1] and Bruhat and Tits [2] have shown that G has a structure theory which is similar in many ways to that of a semisimple Lie group.

Let $\mathbf{P}$ be a minimal parabolic Ω-subgroup of $\mathbf{G}$. Then $\mathbf{P}$ is a split product $\mathbf{P}$ $\mathbf{P} = \mathbf{MN}$ in which $\mathbf{M}$ is connected and reductive and $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$. Let $\mathbf{A}$ be the maximal Ω-split torus in the center of $\mathbf{M}$. For simplicity, we denote the group of Ω-points of each of the above algebraic groups by the corresponding ordinary capital letter.

Let $\overset{\circ}{A}$ be the maximal compact subgroup of A. Then $A/{}^\circ A$ is a free $\mathbf{Z}$-module of rank equal to the Ω-rank of $\mathbf{G}$. We call $\mathfrak{a} = (A/{}^\circ A) \otimes_{\mathbf{Z}} \mathbf{R}$ the Lie algebra of A and write $H: A \longrightarrow \mathfrak{a}$ for the natural map which imbeds $A/{}^\circ A$ as a lattice in $\mathfrak{a}$. To any rational character χ of $\mathbf{A}$ we associate a linear functional on $\mathfrak{a}$ by setting $\log|\chi(a)| = \langle \chi, H(a) \rangle$ $(a \in A)$. The relative Weyl group $W = N_G(A)/Z_G(A)$ operates on A and $\mathfrak{a}$. There is a root system in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ associated to the restriction to $\mathbf{A}$ of the adjoint representation of $\mathbf{G}$. Choosing a W-invariant scalar product on $\mathfrak{a}$, we regard this root system as a subset of $\mathfrak{a}$. Let N correspond to a set of positive roots and let $\overline{N}$ be the Ω-points of the unipotent radical of the opposite parabolic subgroup, corresponding to the negative roots. Write $\mathfrak{a}^+$ [respectively, $+\,\mathfrak{a}$] to denote the (closed) positive chamber in $\mathfrak{a}$ [respectively, the cone consisting of nonnegative linear combinations of the positive roots]. The mapping H extends to M; denoting the kernel of H in M as ${}^\circ M$, we see that H also imbeds $M/{}^\circ M$ as a lattice in $\mathfrak{a}$. Let $M^+ = H^{-1}(\mathfrak{a}^+)$ and $+M = H^{-1}(+\,\mathfrak{a})$.

We choose a particularly "good" maximal compact subgroup K of G (i.e.

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corresponding to the origin in an apartment of the building of G and such that $K \supset B \supset P \cap K = P \cap B$, B a certain Iwahori subgroup of G), the existence of which is guaranteed by Bruhat's and Tits' theory, and recall that G has, with respect to K, both a Cartan decomposition $G = KM^+K$ and an Iwasawa decomposition $G = NMK$. We have the bijections $K\backslash G/K \longleftrightarrow M^+/{}^\circ M$ and $N\backslash G/K \longleftrightarrow M/{}^\circ M$.

For $S \subset \mathfrak{A}$, we write $C(S)$ to denote the convex closure of S and $W \cdot S$ to denote the Weyl group orbit of S. Our main result (cf. Theorem 4.1 of [4]) is

THEOREM. *Let $m, m' \in M$. Then the double cosets KmK and $Nm'K$ intersect if and only if $H(m') \in C(W \cdot H(m))$.*

Since $0 = H(1) \in C(W \cdot x)$ for all $x \in \mathfrak{a}$, we have

COROLLARY. $G = KNK$.

The following simple geometric lemma translates "only if" into a result [2, 4.4.4(iii)] of Bruhat-Tits.

LEMMA 1. *Let $x \in \mathfrak{a}^+$ and set $\mathfrak{a}(x) = \{y \in \mathfrak{a}^+ \mid x - y \in +\mathfrak{a}\}$. Then $W \cdot \mathfrak{a}(x) = C(W \cdot x)$.*

It follows by inspection of the defining integrals that the zonal spherical functions on G, evaluated at fixed $m \in M$ and regarded as functions on $\mathfrak{a}$ (or rather on $\mathfrak{a}^*$), may be interpreted as generating functions for the measures of the intersections $KmK \cap Nm'K$ $(m' \in M)$. The Weyl group invariance of such a function on $\mathfrak{a}^*$ [5] or, what is the same, the equivalence of all principal series representations of G in a Weyl group orbit [3, §8] implies

LEMMA 2. *Let $m, m' \in M$. Then KmK intersects $N'm'K$ for $N = N'$ if and only if the same is true for some (i.e. for every) N' conjugate to N under W.*

The "if" part of our theorem is an easy consequence of the following structural result.

LEMMA 3. *Let B and $\bar{N}$ be as above. Then $B\bar{N}K = B + MK$.*

Details and proofs will appear elsewhere.

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