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IMAGES OF HOMOGENEOUS VECTOR BUNDLES AND VARIETIES OF COMPLEXES

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Let G be a connected algebraic group with a given representation on a vector space V. Let W be a subspace of V. I propose to study the union of all the translates of W by G, $G \cdot W$.

Let P be a subgroup of G that stabilizes W. Let $X \longrightarrow G/P$ be the homogeneous vector bundle over G/P, associated to the representation of P on W. Explicitly

$$X = \{(g, w) \in G \times W \text{ modulo } (g, w) \sim (gp^{-1}, pw) \text{ for } p \in P\}.$$

The representation $G \times V \longrightarrow V$ induces a morphism $f: X \longrightarrow V$. The image of f is $G \cdot W$.

THEOREM. *Assume G/P is complete. Then $G \cdot W$ is a closed subvariety of V. Furthermore, if the characteristic of the ground field is zero, and if the actions of G on V and of P on W are completely reducible, then $G \cdot W$ is a normal Cohen-Macaulay variety, and if f is birational, then $G \cdot W$ has rational singularities.*

The proof of this theorem uses the Borel-Weil-Bott theorem on the cohomology of homogeneous vector bundles [1] together with some facts surrounding the theory of rational resolutions [5].

The application that I have in mind for this theorem is the study of the singularities of the varieties of complexes introduced by Buchsbaum and Eisenbud [2].

I will first state what these varieties are. Let $K^0, \ldots, K^n$ be a sequence of vector spaces. Let V be the direct sum of $\mathrm{Hom}(K^0, K^1), \ldots, \mathrm{Hom}(K^{n-1}, K^n)$. A point a in V is denoted $(a_1, \ldots, a_n)$, where $a_i \in \mathrm{Hom}(K^{i-1}, K^i)$. A point a in V represents a complex if $a_{i+1} \circ a_i = 0$ for $0 < i < n$. The rank of a is the sequence of integers, (rank $a_1, \ldots,$ rank a_n), where rank b is the dimension of the image of the homomorphism b. If $(m_1, \ldots, m_n)$ is the rank of a complex, then $m_1 \leqslant \dim K^0$, $m_n \leqslant \dim K^n$, and $m_i + m_{i+1} \leqslant \dim K^i$ for $0 < i < n$. Conversely, any such sequence is the rank of a complex. Let M be the set of such sequences.

If $m \in M$, define the variety of Buchsbaum and Eisenbud, B-E(m), to be the variety of complexes a, such that rank $a_i \leqslant m_i$ for $1 \leqslant i \leqslant n$.

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THEOREM. *In characteristic zero, a Buchsbaum-Eisenbud variety of complexes is a normal Cohen-Macaulay variety and has rational singularities.*

Of course, this last theorem should hold in any characteristic. The ideal of functions vanishing on B-E(m) should be generated by the quadric relations, corresponding to the complex condition, and the determinants, corresponding to the rank conditions. One should be able to prove that these functions generate a prime ideal by the inductive method used by Eagon and Hochster [3] where they settle the case when $n = 1$ (see also [4]).

In the rest of this note, I will sketch how the first theorem implies the second. I plan to publish the first theorem separately.

Let $G = \mathrm{GL}(K^1) \times \ldots \times \mathrm{GL}(K^n)$. Consider a sequence $L = (L^i, \ldots, L^n)$, where L^i is a subspace of dimension m_i of K^i. Let $W(L)$ be the subspace of V consisting of a's such that the image of $a_i \subset L^i \subset$ the kernel of a_{i+1} whenever either statement makes sense. Let J be a fixed sequence of subspaces as above. Let $W = W(J)$. Let P be the product of the stabilizer of J^i in $\mathrm{GL}(K^i)$ for $1 \leqslant i \leqslant n$. P stabilizes W. The homogeneous vector bundle X is the totality of all pairs L and a such that $a \in W(L)$.

G/P is the space of L's, which is a product of Grassmannian. Hence, G/P is complete. The mapping f sends (L, a) onto a. The image is clearly all of B-E(m). If a point $a \in$ B-E(m) satisfies the rank conditions exactly, there is a unique pair (L, a) in X. In fact, L^j must be the image of a_j. The morphism f is birational because it is an isomorphism near any such "general" point of B-E(m).

It remains to verify to complete reducibility assumptions. Assume the characteristic is zero. Then, as G is a reductive group, any of its representations are completely reducible. As for P acting on W, note that $a \in W$ is determined by the induced homomorphisms

$$K^0 \longrightarrow J^1, K^1/J^1 \longrightarrow J^2, \ldots, K^{n-1}/J^{n-1} \longrightarrow J^n.$$

Thus, the action of P on W is equivalent to the action of its quotient $\mathrm{GL}(J^1) \times \mathrm{GL}(K^1/J^1) \times \ldots \times \mathrm{GL}(J^n)$, which is again reductive.

Therefore, we have not only found a pleasant resolution of the singularities of B-E(m), but we conclude that the first theorem implies the second.

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