linear operator from V into R^n. Then the problem

$$(14) \qquad P:\inf_{u,p}\int_\Omega f[x,u(x),p(x)]\,dx, \qquad u=Gp, \quad p\in[L^1(\Omega)]^n,$$

has at least one solution. If f satisfies all the conditions above except the convexity in ξ, it is true that the problem

$$(15) \qquad PR \equiv P^{**}:\int_\Omega f^{**}[x,u(x),p(x)]\,dx$$

has at least one solution and

$$(16) \qquad \min(P^{**})=\inf(P).$$

We can also say that if $(\bar u,\bar p)$ with $\bar u=G\bar p$ is a solution of (PR), then there is a minimizing sequence (u_q,p_q), with $u_q=Gp_q$, for (P) such that $u_q\to\bar u$ a.e. and $p_q\to\bar p$ weakly in L_1 (actually more can be said about the convergence); *any such sequence contains a subsequence $(u_{q'},p_{q'})$ such that $u_{q'}\to\bar u$ and $Gp_{q'}\to G\bar p$ in L_1.*

Chapter X extends and refines many of the results obtained in the preceding chapters. The fundamental problem of the calculus of variations

$$(P) \qquad \inf\int_\Omega f[x,u(x),\operatorname{grad} u(x)]\,dx, \qquad u-u_0\in W_0^{1,\alpha}(\Omega),$$

receives special attention. The chapter concludes with a discussion of results involving the Euler equations and the problems (*PR*) (i.e., (P^{**})).

CHARLES B. MORREY, JR.

Characteristic classes by John W. Milnor and James D. Stasheff, Ann. of Math. Studies No. 76, Princeton Univ. Press, Princeton, New Jersey, 1974, vii+330 pp., $10.00

In 1957 there appeared notes by Stasheff of lectures on characteristic classes by Milnor at Princeton University. These notes are a clear concise presentation of the basic properties of vector bundles and their associated characteristic classes. Since their appearance they have become a standard text regularly used by graduate students and others interested in learning the subject.

The present, long-anticipated book is based on those notes. It follows the order of the notes but is considerably expanded with more detail and discussion. In addition, exercises have been added to almost each section, there are many useful references to the textbooks on algebraic topology that are available now, and there is an epilogue summarizing main developments in the subject since 1957. All of these strengthen the book and make it even more valuable as a text for a course as well as a book that can be read by students on their own. The material covered should be required for doctoral students in algebraic or differential topology and strongly recommended for those in differential geometry.

In the last thirty years the theory of vector bundles has become an important part of the mainstream of modern mathematics. Roughly speaking, a vector bundle of dimension n over a space X is a family of real or complex n-dimensional vector spaces parametrized by X. Characteristic classes are cohomology classes of X associated to such a vector bundle. Thus, characteristic classes occur naturally in the study of vector bundles.

Historically the subject originated in differential geometry. The geometry of a smooth manifold involves various tensor fields on the manifold. Such fields were defined classically in terms of local transformation properties, but, in current terminology, they are interpreted as cross-sections of suitable tensor bundles associated to the tangent bundle of the manifold. These tensor bundles, and the tangent bundle itself, are examples of vector bundles over the manifold, and the concept of vector bundle was introduced primarily to provide a global setting for the constructs of differential geometry in this way.

On a compact oriented smooth manifold the Euler characteristic determines one of the characteristic classes of the tangent bundle. In fact, it equals the value on the fundamental homology class of the manifold of the Euler class, a particular characteristic class of the tangent bundle. It is a classical result that the vanishing of the Euler characteristic is a necessary and sufficient condition for the existence of a nowhere zero tangent vector field on the manifold. This result has generalizations which provide necessary conditions the characteristic classes of a manifold must satisfy if the manifold admits k linearly independent tangent vector fields. The characteristic classes are also applied to obtain necessary conditions for the existence of immersions of the manifold in euclidean space with codimension k.

The definition of vector bundle does not require a smooth manifold; in fact it is natural, and useful, to consider vector bundles over arbitrary topological spaces. Hence the theory of vector bundles is a part of the subject of topology. Consideration of these bundles and their functorial properties has led to K-theories, first examples of generalized cohomology theories (a generalized cohomology theory satisfies all of the Eilenberg-Steenrod axioms for cohomology theory except the dimension axiom). These K-theories and their operations have proved fruitful in various contexts. For example, they were used to solve the problem of finding the number of linearly independent vector fields on the n-sphere. They were also used to formulate the "index theorem" which equates an analytic invariant of certain manifolds with a topological invariant of the manifold. In this way the theory of vector bundles has developed into an important tool, both in topology and in certain parts of analysis.

The book under review is an elegant treatment of the subject. Prerequisite are the standard facts of algebra and point set and algebraic topology such as a typical student might have by the end of a year of graduate study. The main concepts and results are lucidly presented and easily accessible. It is an excellent source for anyone who wants to learn or review the essentials of the subject.

The book is divided into parts treating the various characteristic classes in order. The first ones considered are the Stiefel-Whitney classes associated to real vector bundles. Here they are defined by four axioms, naturality, normalization, nontriviality, and the Whitney product theorem, which is the most crucial and expresses the classes of the Whitney sum of two vector bundles in terms of the classes of the summands. These axioms are then immediately used to obtain some results about nonimmersibility of real projective spaces in euclidean spaces and to show that necessary conditions for a manifold M to be the boundary of a compact manifold are that all the Stiefel-Whitney numbers of M be zero. The latter result is important for the calculation of cobordism groups. These applications show the utility of, and provide motivation for, the Stiefel-Whitney classes.

Having these justifications for consideration of the Stiefel-Whitney classes their uniqueness and existence are established. Uniqueness is proved by direct cohomology calculations in the $\mathbf{Z}/2$ cohomology of the Grassmann manifolds (these being base spaces of the universal bundles). Then existence is proved assuming the existence and standard properties of the Steenrod squaring operations and the Thom isomorphism theorem. (The latter relates the homology and cohomology of a vector bundle to that of its base space and is established later in the book.) This concludes the first portion of the book dealing with the Stiefel-Whitney classes.

Next is the definition of the Euler class of an oriented real vector bundle. It is an integral characteristic class and is defined by using the Thom isomorphism theorem for oriented bundles. Its vanishing is shown to be a necessary condition for the vector bundle to have a nowhere zero cross-section. The theory is then applied to the normal bundle of a smooth manifold embedded in a Riemannian manifold. For the particular case of a Riemannian manifold M embedded as the diagonal in $M \times M$ there results the relation between the Euler class and Euler characteristic of a smooth compact oriented manifold referred to earlier in this review. Other results that follow by considering this diagonal embedding are the Wu formulas which express the Stiefel-Whitney classes of a smooth compact manifold M in terms of the Steenrod squaring operations in M. These results imply that the characteristic classes (thus far defined) of the tangent bundle of a real manifold are homotopy invariants of the manifold.

The original definitions of the characteristic classes in the literature were in terms of obstructions to the existence of families of everywhere linearly independent cross-sections of the vector bundle in question. This portion of the book concludes with a discussion of the relation between these obstruction characteristic classes and the characteristic classes defined earlier in the book.

The next part of the book introduces the Chern classes of a complex vector bundle. These are integral characteristic classes in even dimensions and are defined inductively in terms of the Euler class of the corresponding real vector bundle using the Gysin sequence of this bundle. The important product formula for Chern classes is proved by cohomology calculations in the complex Grassmann manifolds. Next the Pontrjagin classes of a real vector bundle are defined using the Chern classes of the complexification of the vector bundle. These are integral characteristic classes in dimensions divisible by 4. With this, all of the characteristic classes to be considered in the book have been defined.

The rest of the body of the text is devoted to applications of the characteristic classes. First is an application to partial calculation of the cobordism groups. It is shown that the vanishing of all Pontrjagin numbers of a smooth compact oriented manifold are necessary conditions for the manifold to be a boundary. This leads to the study of the oriented cobordism ring and to a calculation of the tensor product of this ring with the rationals.

Next is a proof of the signature theorem using multiplicative sequences. This theorem expresses the signature of a smooth compact oriented manifold in terms of Pontrjagin numbers of M. It is applied to prove that the rational Pontrjagin classes of a smooth compact oriented manifold are piecewise linear invariants of the manifold. In fact, rational Pontrjagin classes are defined for compact rational homology manifolds and these combinatorial Pontrjagin classes are shown to agree with the differentiable ones for smooth manifolds. As one application of this, it is shown that for every dimension $\geqq 8$ there are two smooth simply-connected manifolds having the same homotopy type but not piecewise linearly homeomorphic. Another application is to provide an example of a triangulated 8-dimensional compact manifold having no compatible smooth structure.

In addition to those results dealing directly with characteristic classes the book contains a discussion of various topics of interest in algebraic and differential topology which are relevant to the text. The epilogue is a brief survey of results since the original notes appeared. It has three parts, one devoted to generalizations of the theory to nonsmooth manifolds, one devoted to the theory for smooth manifolds with additional structure, and one devoted to generalized cohomology theories.

As can be seen from this discussion of the contents, there is a lot of mathematics included in the book. It is a valuable and welcome addition to the literature.

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Lie groups, Lie algebras, and their representations, by V. S. Varadarajan, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974, xiii+430 pp., $19.95

Compact Lie groups and their representations, by D. P. Želobenko, Translations of Mathematical Monographs, Vol. 40, American Mathematical Society, Providence, R.I., 1973, viii+448 pp., $35.70

The books by Varadarajan and Želobenko are surprisingly dissimilar. Both are written as detailed mathematical expositions of the theory of representations of compact Lie groups. But Varadarajan's is written with mathematics students in mind, whereas Želobenko's is influenced by the needs of theoretical physicists.